DOI : 10.15613/sijrs/2017/v4i1/172389
ISSN (Print): 2349-4778
ISSN (Online): 2349-8900
ScieXplore: International Journal of Research in Science, *Vol 4(1), 7–12, January-June 2017*

Oscillation of Impulsive Hyperbolic Differential Equations with Distributed Delay

V. Sadhasivam[1], T. Kalaimani[2] and K. Logaarasi[3]

PG and Research Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram – 637 401, Namakkal Dt., Tamil Nadu, India; ovsadha@gmail.com, kalaimaths4@gmail.com and logajoni@gmail.com

Abstract

The present effort deals about oscillation of solutions of impulsive hyperbolic differential equations with distributed deviating arguments. Sufficient conditions are obtained for the oscillation of solutions using impulsive differential inequalities and integral averaging scheme with boundary condition. Example is provided to illustrate the obtained results.

Keywords: Distributed Delay, Impulse, Oscillation, Partial Differential Equations.
2010 Mathematics Subject Classification: 35B05, 35L25, 35R10, 35R12

1. Introduction

In recent years considerable attention has been given to the study of oscillation and nonoscillation results with continuous distributed deviating arguments[4],[5],[8-13],[15-17]. The study of impulsive partial differential equations is motivated by having many applications in population models[4],[7], single species growth[6], quenching problems[3] and various scientific models[18],[19] with the boundary conditions of the type Dirichlet, Neumann and Robin. The current research focus on oscillation of the following impulsive partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = a(t)\Delta u(x,t) + b(t)\Delta u(x,\tau(t))$$
$$-\sum_{i=1}^{n} r_i(x,t)u(x,\sigma_i(t))$$
$$+\int_c^d q(x,t,\xi) f\left(u\left(x,v(t,\xi)\right)\right)d\eta(\xi), t \neq t_k,$$
$$(x,t)\in \Omega \mathrm{X}(0,+\infty) \equiv G,$$
$$u(x,t_k^+) = (1+\alpha_k)u(x,t_k)$$
$$u_t(x,t_k^+) = (1+\beta_k)u_t(x,t_k),\ k=1,2,\ldots. \quad (1)$$

with

$$u=0, \quad (x,t)\in \partial\Omega \mathrm{X}(0,+\infty), \quad (2)$$

where Δ is the Laplacian in $\mathbb{R}^N$ and Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$.

Now we present a set of conditions that will be assumed throughout the paper.

$(\boldsymbol{H}_1)$ $a(t), b(t) \in PC([0,+\infty),[0,+\infty))$, where PC represents the set of functions which are piecewise continuous with discontinuous of the first kind in $t = t_k$ and left continuos at $t = t_k$

$(\boldsymbol{H}_2)$ $r_i(x,t)\in C(\bar{\Omega}\mathrm{X}[0,+\infty),[0,+\infty))$,
$D_i(t) = \min_{x\in\Omega} r_i(x,t)$, $q(x,t,\xi)\in C(\bar{\Omega}\mathrm{X}\mathbb{R}^+\mathrm{X}[c,d],\mathbb{R}^+)$,
$Q(t,\xi) = \min_{x\in\Omega} q(x,t,\xi)$, $f(u)\in C(\mathbb{R},\mathbb{R})$

**Author for correspondence*

is convex in $\mathbb{R}^+, uf(u) > 0$ and $\frac{f(u)}{u} \geq \in > 0$ for $u \neq 0$.

$(\boldsymbol{H}_3)\, \tau(t) \in C([0,+\infty), \mathbb{R}), \lim_{t\to\infty} \tau(t) = +\infty,$

$\sigma_i(t) \in C([0,+\infty), \mathbb{R}), \lim_{t\to+\infty} \sigma_i(t) = +\infty, \sigma(t) = \max_{1\leq i\leq n} \sigma_i(t),$
$i = 1,2,\ldots,n, v(t,\xi) \in C(\mathbb{R}^+ \mathrm{X}[c\ d]\mathbb{R}),$
$v(t,\xi) \leq t\ for \xi \in [c,d]$ *and* $v(t,\xi)$ is nondecreasing with respect to t and ξ respectively. More over $\liminf_{t\to\infty, \xi\in[c,d]} v(t,\xi) = +\infty, \eta(\xi) : [c,d] \to \mathbb{R}$ is nondecreasing and also the integral in (1) is a stieltjes integral".

$(\boldsymbol{H}_4)$ $u(x,t), u_t(x,t) \in PC(G,\mathbb{R}), \alpha_k > -1, \beta_k > -1, \alpha_k < \beta_k,$ the sequence t_1 is a fixed strictly increasing sequence of positive real numbers with $t_k \to \infty$ as $k \to \infty$.

2. Preliminaries

We begin with definitions, known results, notations and Lemma which arerequired throughout this paper.

Definition 2.1

By a solution of (1)-(2) we mean a function u *such that*

$$u \in C^2(\overline{\Omega}\mathrm{X}[w_1,+\infty), \mathbb{R}) \cap C(\overline{\Omega}\mathrm{X}[w_2,+\infty), \mathbb{R})$$

that satisfies (1), where

$$w_1 = \min\left\{0, \min_{\xi\in[c,d]}\left\{\inf_{t\geq 0} v(t,\xi)\right\}\right\},$$

$$w_2 = \min\left\{0, \inf_{t\geq 0} \tau(t), \min_{1\leq i\leq n}\left\{\inf_{t\geq 0} \sigma_i(t)\right\}\right\}.$$

Now with this definition of solution, we can precisely define what we mean by oscillation.

Definition 2.2

A nontrivial solution u is said to be oscillatory in G *if for each* $l > 0$, *there exists a point* $(x_0, t_0) \in \Omega\mathrm{X}[l,+\infty)$ *such that* $u(x_0,t_0) = 0$ *holds.*

It is identified that [14], the least eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\Delta\omega(x) + \lambda\omega(x) = 0, \quad in\ \Omega,$$
$$\omega(x) = 0, \quad on\ \ \partial\Omega,$$

and the consequent eigen function $\varphi(t) > 0$ in Ω

For each positive solution $u(x,t)$ of (1),(2) we define the functions

$$A(t) = K_\varphi \int_\Omega u(x,t)\varphi(x)\,dx,$$

where

$$K_\varphi = \left(\int_\Omega \varphi(x)\,dx\right)^{-1}, |\Omega| = \int_\Omega dx, E(t) = \in \int_c^d Q(t,\xi)\,d\eta(\xi).$$

Lemma 2.3.[2]

Suppose that $y(t) \in C^2([t_0,\infty), \mathbb{R})$ *and that* $y(t) > 0, y'(t) > 0$ *and* $y''(t) \leq 0$ *for* $t \geq t_0 > 0$. (3) *Then for any* $\lambda_1 \in (0,1)$, *there exists a number* $t_1 > t_0$ *such that* $y(t) \geq \lambda_1 t y(t)$ *for* $t \geq t_1$. *(4)*

In Section 3, we discuss the oscillation of the problem (1) - (2) in detail, and in Section 4, an example is presented to verify main results.

3. Main Results

Theorem 3.1 *Assume that conditions* (H_1) *-*(H_4) *hold and that every solution* $u(x,t)$ *of (1),(2) is oscillatory in G, if the impulsive delay differential inequality*

$$A''(t) + \sum_{i=1}^n D_i(t) A(\sigma_i(t)) - E(t) A(v(t,\xi))$$

$$\leq 0,\ t \neq t_k,\ t \geq t_1,$$

$$A(t_k^+) = (1+\alpha_k) A(t_k)$$

$$A'\left(t_k^+\right)=\left(1+\beta_k\right)A'\left(t_k\right), k=1,2,\ldots \quad (5)$$

has no eventually positive solutions.

Proof.

Let $u(x,t)>0$ be a non-oscillatory solution. Then there exists a $t_1>t_0>0$ such that $\tau(t)\geq 0$, $\sigma_i(t)\geq 0$ and $v(t,\xi)\geq 0$ for $(t,\xi)\in [t_1,+\infty)X[c,d]$, we get that $u(x,\tau(t))>0, for\ (x,t)\in\Omega X(t_1,+\infty)$, $u(x,\sigma_i(t))>0, for (x,t)\in\Omega X(t_1,+\infty),\ i=1,2,..n$

and $u(x,v(t,\xi))>0$, for $(x,t,\xi)\in\Omega X(t_1,+\infty)X[c,d]$.

Multiply Equation (1) by $K_\varphi\varphi(x)>0$ and integrate over with respect to x, we obtain

$$\frac{\partial^2}{\partial t^2}\left[\int_\Omega K_\varphi u(x,t)\varphi(x)dx\right]$$

$$=a(t)\int_\Omega K_\varphi\Delta u(x,t)\varphi(x)dx+b(t)\int_\Omega K_\varphi\Delta u(x,\tau(t))\varphi(x)dx$$

$$-\sum_{i=1}^{n}\int_\Omega r_i(x,t)K_\varphi u(x,\sigma_i(t))\varphi(x)dx$$

$$+\int_\Omega\int_c^d f\left(u\left(x,v(t,\xi)\right)\right)K_\varphi q(x,t,\xi)$$

$$\varphi(x)d\eta(\xi)dx. \quad (6)$$

Appling Green's formula and Equation (2), we get that

$$K_\varphi\int_\Omega\varphi(x)\Delta u(x,t)dx=-\lambda_0 A(t).$$

$$\therefore K_\varphi\int_\Omega\varphi(x)\Delta u(x,t)dx\leq 0 \quad (7)$$

and

$$K_\varphi\int_\Omega\varphi(x)\Delta u(x,\tau(t))dx=-\lambda_0 A(\tau(t)). \quad (8)$$

$$\therefore K_\varphi\int_\Omega\varphi(x)\Delta u(x,\tau(t))dx\leq 0.$$

Here dS is surface component on $\partial\Omega$. Furthermore applying Jensen's inequality for convex functions and using the assumptions in (H_2), we get that

$$\int_\Omega\int_c^d f\left(u\left(x,v(t,\xi)\right)\right)K_\varphi q(x,t,\xi)\varphi(x)d\eta(\xi)dx$$

$$\geq\int_c^d Q(t,\xi)\in\int_\Omega K_\varphi u\left(x,v(t,\xi)\right)\varphi(x)dxd\eta(\xi)$$

$$\geq\in\int_c^d A\left(v(t,\xi)\right)Q(t,\xi)d\eta(\xi)", \quad (9)$$

where $A\left(v(t,\xi)\right)=\int_\Omega K_\varphi u\left(x,v(t,\xi)\right)\varphi(x)dx.$

Combining (6)-(9), we get that

$$A''(t)\leq-\sum_{i=1}^{n}\int_\Omega D_i(t)K_\varphi u(x,\sigma_i(t))\varphi(x)dx$$
$$+\in\int_c^d A\left(v(t,\xi)\right)Q(t,\xi)d\eta(\xi),$$

$$A''(t)+\sum_{i=1}^{n}A(\sigma_i(t))D_i(t)-A\left(v(t,\xi)\right)E(t)\leq 0,\ t\neq t_k,\ t\geq t_1.$$

Where $E(t)=\in\int_c^d Q(t,\xi)d\eta(\xi)$.

Also, multiply (1) by $K_\varphi\varphi(x)>0$, integrate over Ω, and from (H_4) we obtain

$$\int_\Omega K_\varphi u\left(x,t_k^+\right)\varphi(x)dx$$

$$=\left(1+\alpha_k\right)\int_\Omega K_\varphi u\left(x,t_k\right)\varphi(x)dx,$$

For $t=t_k$,

$$A\left(t_k^+\right)=\left(1+\alpha_k\right)A\left(t_k\right)$$

$$A'\left(t_k^+\right)=\left(1+\beta_k\right)A'\left(t_k\right),\ k=1,2,\ldots.$$

Therefore $A(t)$ is an eventually positive solution of (5). This disagrees the hypothesis.

Theorem 3.2. *Suppose that (H_1) - (H_4) hold. If for every number* $\lambda_1,\lambda_2 \in (0,1)$,

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \prod_{t_0\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)\left[\sum_{i=1}^{n}\lambda_1\sigma_i(s)D_i(s)-\lambda_2E(s)v(s,\xi)\right]ds>1, \quad (10)$$

then every solution $u(x,t)$ *of Equations (1), (2) is oscillatory in G.*

Proof. On the contrary, let $u(x,t)$ be a nonoscillatory solution of Equations (1), (2) which we assume to be positive. Now we can use

$$A''(t)+\sum_{i=1}^{n}A(\sigma_i(t))D_i(t)-A(v(t,\xi))E(t)\le 0,\quad t\ge t_1.$$

By Lemma 2.4

$$A(\sigma_i(t))\ge \sigma_i(t)\lambda_1 A'(\sigma_i(t)) \text{ and}$$

$$A(v(t,\xi))\ge v(t,\xi)\lambda_2 A'(v(t,\xi)),\quad t\ge t_1.$$

From this,

$$\begin{aligned}&A''(t)+\sum_{i=1}^{n}D_i(t)\sigma_i(t)\lambda_1A'(\sigma_i(t))\\&-E(t)v(t,\xi)\lambda_2A'(v(t,\xi))\le 0, t\ge t_1.\end{aligned} \quad (11)$$

Define

$B(t)=\prod_{t_0\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}A(t).$ In fact, $B(t)$ is continuous on every $[t_k,t_{k+1}]$ and in consideration of $B(t_k^+)\le\left(\frac{1+\beta_k}{1+\alpha_k}\right)B(t_k),$

it follows for $t\ge t_0$, the following inequality does not have

eventually positivesolution when inequality (11) does not have the same solution.

$$B''(t)+\sum_{i=1}^{n}\lambda_1\sigma_i(t)D_i(t)B'(\sigma_i(t))$$

$$-E(t)v(t,\xi)\lambda_2B'(v(t,\xi))\le 0, t\ge t_1.$$

Where

$$B(t_k^+)=\prod_{t_0\le t_j\le t_k}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}A(t_k^+)$$

$$=\prod_{t_0\le t_j\le t_k}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}A(t_k)=B(t_k),$$

$$B(t_k^-)=\prod_{t_0\le t_j\le t_{k-1}}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}A(t_k^-)$$

$$=\prod_{t_0\le t_j<t_k}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}A(t_k)=B(t_k).$$

Which gives that, $B(t)$ is continuous on $[t_0,+\infty)$. Then we get that

$$\begin{aligned}&\prod_{t_0\le t_k<t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-2}A''(t)\\&+\sum_{i=1}^{n}\lambda_1\sigma_i(t)D_i(t)\prod_{t_0\le t_k<t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}\\&A'(\sigma_i(t))-E(t)v(t,\xi)\lambda_2\prod_{t_0\le t_k<t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}\\&A'(v(t,\xi))\le 0.\end{aligned}$$

Integrate the previous inequality between $\sigma(t)$ and t, we have

$$\begin{aligned}&\prod_{t_0\le t_k<t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-2}\left[A'(t)-A'(\sigma(t))\right]\\&+\int_{\sigma(t)}^{t}\sum_{i=1}^{n}\lambda_1\sigma_i(s)D_i(s)\prod_{t_0\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}A'(\sigma_i(s))ds\\&-\int_{\sigma(t)}^{t}\lambda_2E(s)v(s,\xi)\prod_{t_0\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}A'(v(s,\xi))ds\\&\le 0,\quad t\ge t_1.\end{aligned}$$

Therefore,

$$\int_{\sigma(t)}^{t}\prod_{t_0\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)\left[\sum_{i=1}^{n}\lambda_1\sigma_i(s)D_i(s)-\lambda_2E(s)v(s,\xi)\right]ds$$

$$\le 1-\frac{A'(t)}{A'(\sigma(t))}<1,$$

and hence

$$\begin{aligned}&\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\prod_{t_0\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)\\&\left[\sum_{i=1}^{n}\lambda_1\sigma_i(s)D_i(s)-\lambda_2E(s)v(s,\xi)\right]ds\le 1,\end{aligned} \quad (12)$$

which made a contradiction with (10). The proof when $u(x,t) < 0$ is similar and will be omitted here.

4. Example

The present section contains an example to point up the key results established in Section 3.

Example 4.1. *Consider the following equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{5}{2}\Delta u(x,t) + \Delta u(x,t-3\pi/2)$$
$$-\frac{5}{2}u(x,t-\pi) - \int_{-\pi/2}^{0} u(x,t+\xi)\,d\xi,$$
$$t \neq t_k, t > 1.$$
$$u(x,t_k^+) = (1+\alpha_k)u(x,t_k)$$
$$u_t(x,t_k^+) = (1+\beta_k)u_t(x,t_k),\ k = 1,2,\ldots. \quad (13)$$

for $(x,t) \in (0,\pi)X(0,+\infty)$, *with*

$$u(0,t) = u(\pi,t) = 0. \quad (14)$$

Here

$$\Omega = (0,\pi), a = \frac{5}{2}, b(t) = 1, \tau(t) = t - \frac{3\pi}{2},$$
$$n = 1, D = \frac{5}{2}, \sigma(t) = t - \pi, Q(t,\xi) = 1,$$
$$f(u) = u, v(t,\xi) = t + \xi, \alpha_k = \frac{1}{2^k},$$
$$\beta_k = 2^k, \in = 1 \text{ and } E(s) = \frac{\pi}{2}.$$

Also, we see from the above assumption that the hypotheses (H_1)-(H_4) hold, moreover

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right) = \int_{1}^{+\infty}\prod_{1<t_k<s}\left(\frac{1+2^k}{1+(1/2^k)}\right)ds$$
$$= \int_{1}^{t_1}\prod_{1<t_k<s}2^k\,ds + \int_{t_1^+}^{t_2}\prod_{1<t_k<s}2^k\,ds + \int_{t_2^+}^{t_3}\prod_{1<t_k<s}2^k\,ds + \ldots$$

Now, the condition (10) reads,

$$\limsup_{t\to\infty}\int_{t-\pi}^{t}\left(\prod_{t_0\le t_k<s}2^k\right)\left[\frac{5}{6}(s-\pi) + \frac{\pi}{8}(s+\xi)\right]ds$$
$$> 1. \quad (15)$$

Thus the provisions of the Theorem 3.2 are fulfilled and hence all the solutions of Equations (13)-(14) are oscillatory in G. Actually $u(x,t) = \sin x \cos t$ is one such solution.

4. References

1. Bainov D, Minchev E. Estimates of solutions of impulsive parabolic equations and applications to the population dynamics. Publ Math. 1996; 40:85–94. Doi: https://doi.org/10.5565/PUBLMAT_40196_06.
2. Baotong C, Lalli BS, Yuanhong Y. Forced oscillations of hyperbolic equations with deviating arguments. Acta Mathematicae Applicatae Sinica. 1995; 4:369–77.
3. Chan CY, Ke L, Remarks on impulsive quenching problems. Proc Dynam. Systems Appl. 1994; 1:59–62.
4. Deng L, Ge W, Wang PG. Oscillation of functional parabolic differential equations under the Robin boundary condition. Indian J Pure Appl Math. 2001; 32(4):509–14.
5. Du LL, Fu W, Fan MS. Oscillatory solutions of delay hyperbolic system with distributed deviating arguments. Appl Math Comput. 2004; 154:521–29. Doi: https://doi.org/10.1016/S0096-3003(03)00732-X.
6. Erbe LH, Freedman HI, Liu XZ, Wu JH. Comparison principles for impulsive parabolic equations with applications to models of single species growth. J Austral Math Soc Ser. 1991; B32:382–400. Doi: https://doi.org/10.1017/S033427000000850X.
7. Gao W, Wang J. Estimates of solutions of impulsive parabolic equations under Neumannboundary condition. J Math Anal Appl. 2003, 283:478–90. Doi: https://doi.org/10.1016/S0022-247X(03)00275-0.
8. Liu Y, Zhang J, Yan J. Oscillation properties of higher order partial differential equations with distributed deviating arguments. Discrete Dyn Nat Soc. 2015; 2015:1–9. Doi: https://doi.org/10.1155/2015/206261
9. Luo LP, Gao ZH, Ouyang ZG. Oscillation of nonlinear neutral parabolic partial differential equations with continuous distributed delay. Math Appl. 2006; 19:651–55.
10. Sadhasivam V, Raja T, Kalaimani T. Oscillation of nonlinear impulsive neutral functional hyperbolic equations with damping. International Journal of Pure and Applied Mathematics. 2016; 106(8):187–197.
11. Sadhasivam V, Raja T, Kalaimani T. Oscillation of impulsive neutral hyperbolic equations with continuous distributed deviating arguments, Global Journal of Pure and Applied Mathematics. 2016; 12(3):163–167.

12. Shoukaku Y. Forced oscillatory result of hyperbolic equations with continuous distributed deviating Arguments. Appl Math Lett. 2011; 24:407–411. Doi: https://doi.org/10.1016/j.aml.2010.10.012
13. Tanaka S, Yoshida N. Forced oscillation of certain hyperbolic equations with continuous distributed deviating arguments. Ann Polon Math. 2005; 85:37–54. Doi: https://doi.org/10.4064/ap85-1-4
14. Vladimirov VS. Equations of mathematical physics. Moscow: Nauka; 1981.
15. Wang PG. Oscillatory criteria of nonlinear hyperbolic equations with continuous deviating arguments. Appl Math Comput. 1999; 106:163–169. Doi: https://doi.org/10.1016/S0096-3003(98)10110-8
16. Wang PG, Wu Y, Caccetta L. Forced oscillation of a class of neutral hyperbolic differential equations. J Comput Appl Math. 2005; 177:301–308. Doi: https://doi.org/10.1016/j.cam.2004.09.021
17. Wang PG, Zhao J, Ge W. Oscillation criteria of nonlinear hyperbolic equations with functional arguments. Comput Math Appl. 2000; 40:513–21. Doi: https://doi.org/10.1016/S0898-1221(00)00176-0
18. Wu JH. Theory and applications of partial functional differential equations. New York: Springer; 1996. Doi: https://doi.org/10.1007/978-1-4612-4050-1
19. Yoshida N. Oscillation theory of partial differential equations. Singapore: World Scientific; 2008. Doi: https://doi.org/10.1142/7046