

Prime and Maximal Ideals in Ternary Semigroups

P. Bindu¹, Y. Sarala^{2*}, D. Madhusudhana Rao³ and G. Jaya Lalitha¹

¹Department of Mathematics, KL University, Guntur - 522502, Andhra Pradesh, India

²Faculty of Mathematics, Department of Mathematics, KL University, Guntur – 522502, Andhra Pradesh, India; saralayella1970@gmail.com

³Faculty of Mathematics, Department of Mathematics, VSR and NVR College, Tenali - 522201, Andhra Pradesh, India; dmrmaths@gmail.com; jayalalitha.yerrapothu@gmail.com

Abstract

This paper deals with prime and maximal ideals in ternary semigroup. In this paper we shall study the intersection of all prime ideals and intersection of all maximal ideals in a (non-commutative) ternary semigroup T . In particular we give a rather general necessary and sufficient condition in order that the set of all maximal ideals coincides with the set of all prime ideals.

Keywords: Maximal Ideal of a Ternary Semigroup, Prime Ideal

1. Introduction

The relations between maximal and prime ideals in commutative rings are well known. If R is a ring, denote by N_0 the set of all nilpotent elements $s \in R$. We recall the following results.

- In any commutative ring the intersection of all the prime ideals is N_0 .
- In any commutative ring with identity element any maximal ideal is prime.
- If R is a commutative ring with identity element satisfying the descending chain condition every prime ideal of R is maximal.

Note explicitly that in a ring without an identity element maximal ideals need not be small prime and prime ideals need not be maximal [For this result, where some notions concerning semigroups are involved]⁸.

There are some reasons to have in mind the following analogy:

Rings with an identity element $\leftrightarrow$ ternary semigroups satisfying $T^3 = T$.

Rings without an identity element $\leftrightarrow$ ternary semigroups in which $T^3 \neq T$.

2. Definition 1

A non-empty ideal Q of a ternary semigroup T is said to be prime if $ABC \subset Q$ implies that $A \subset Q$ or $B \subset Q$ or $C \subset Q$; A, B, C being ideals of T .

Remark: There is an analogous definition, an ideal Q is completely prime if $abc \in Q$ implies that $a \in Q$ or $b \in Q$ or $c \in Q$; $a, b, c \in T$. An ideal which is completely prime. But the converse need not be true. These concepts coincide if T is commutative. In this paper we consider prime ideals in the sense of our definition prime ideals in the case of a ternary semigroup have been thoroughly studied in⁷.

Example 1: The ternary semigroup T itself is always a prime ideal of T . But T need not have prime ideals $\neq T$.

Let e.g. $T = \{0, x, x^2, \dots, x^{m-1}\}$, $m \geq 2$ be a ternary semigroup with zero in which $x^m = 0$. Any ideal $\neq T$ is of the form $I_p = \{a^p, \dots, a^{m-1}, 0\}$, $3 \leq p \leq m$. Since we have $I_p \not\subset I_{p-1}$ and $I_{p-1} I_{p-1} I_{p-1} \subset I_p$, I_p is not a prime ideal of T .

3. Definition 2

An ideal M of T is known as a maximal if M is a proper ideal of T and is not properly contained in any proper ideal of T .

There are known some results concerning the existence of maximal ideals^{1,2}. We shall not deal explicitly with these questions.

Example 2: The following example shows that a prime ideal need not be necessarily embeddable in a maximal ideal.

Denote by T_1 the multiplicative ternary semigroup of numbers x satisfying $0 \leq x \leq 1$. Adjoin an element a and consider the set $S = T_1 \cup \{a\}$. Define in S a commutative multiplication $\circ$ by

$$xoyoz = \begin{cases} xyz & \text{if } x, y, z \in T_1 \\ 0 & \text{if } x \in T_1, y = z = a \\ 0 & \text{if } y \in T_1, x = z = a \\ 0 & \text{if } z \in T_1, x = y = a \\ a & \text{if } x = y = z = a \end{cases}$$

Then S is ternary semigroup and $S^3 = S$, S contains a unique maximal ideal, namely T_1 . The set $I = \{0, a\}$ is a prime ideal of S . If $T_\alpha = \{x/0 \leq x < \alpha < 1\}$, then $\{0, a, T_\alpha\}$ is an ideal containing I , but clearly there does not exist a maximal ideal of S containing I .

In the following when speaking about maximal ideals we suppose, of course, that maximal ideal exists.

4. Theorem 1

If T is a ternary semigroup with $T = T^3$, then every maximal ideal of T is prime ideal of T .

Proof: Let M be a maximal ideal of T . Denote $T - M = P$. We first prove $P \subset P^3$, we have

$$T = (M \cup P)^3 = M^3 \cup MMP \cup MPM \cup PMM \cup MPP \cup PMP \cup PPM \cup P^3 \subset M \cup P^3.$$

Since $M \cap P = \emptyset$, we have $P \subset P^3$.

Let now A, B, C be three ideals of T , none of them contained in M such that $ABC \subset M$. Since $A \not\subset M$ and M is maximal, we have $A \cup M = T$, hence $P \subset A$. By the same argument $P \subset B$ and $P \subset C$. Hence $P^3 \subset ABC, P \subset ABC$. This contradicts $ABC \subset M$.

Remark: If $T^3 \neq T$, then Theorem 1 does not hold. For, let $a \in T - T^3$. Then $M = T - \{a\}$ is a maximal ideal of T and it is certainly not prime, since $T^3 \subset M$ while $T \not\subset M$. But we can prove the following:

5. Theorem 2

If M is a maximal ideal of a ternary semigroup T such that $T - M$ contains either more than one element or an idempotent then M is a prime ideal of T .

Proof: We shall use the following well known fact: If M is a maximal ideal of T , then the difference ternary semigroup T/M is simple and if T/M contains more than one element, then T/M cannot be nilpotent. Write again $T = M \cup P, M \cap P = \emptyset$. Let A, B, C be three ideals of T none of them contained in M such that $ABC \subset M$ we again we have $A \cup M = B \cup M = C \cup M = T$, hence $P \subset A, P \subset B, P \subset C$ and $P^3 \subset ABC$; therefore $P^3 \subset M$. This would imply that T/M is nilpotent, which is impossible in both cases considered in the statement of our theorem.

Example 3: The following example serves to clarify the situation. Let S be the multiplicative ternary semigroup of numbers $\{x/0 \leq x \leq \frac{1}{2}\}$ and G commutative group. Define $T = S \cup G$ a multiplication $\circ$ by $x \circ y = 0$ if $x \in T, y \in G$ while the product in S and G remain the old ones. Then T is a ternary semigroup with $S^3 \neq S$. Here S is a maximal ideal which is prime. There is an infinity of further maximal ideals, namely the sets $M_t = T - \{t\}$ where t is any element with $\frac{1}{4} \leq t \leq \frac{1}{2}$, none of them being a prime ideal of T .

Example 4: Let T be the set of all integers ≥ 3 , the multiplication being the ordinary multiplication of numbers. The sets $Q_p = \{p, 3p, \dots\}$ ($p = \text{Prime}$) are prime ideals of T and clearly the $\bigcap Q_p$ (where p runs through all primes) is empty.

[Note that any union of the type $\bigcup_{p \in A} Q_p$ (A a subset of the set of all primes) is a prime ideal of T .]

In contradiction to this we shall see in Theorem 3 that the intersection of all maximal ideals of any ternary semigroup is always non empty. In our example the maximal ideals are the sets $M_p = T - \{p\}$ and we have $\bigcap_p M_p = T^3$.

We intend to clarify under which conditions prime ideals are maximal ideals. To this we first prove the following crucial theorem.

6. Theorem 3

Let $\{M_\alpha/\alpha \in A\}$ be the set of all different maximal ideals of a ternary semigroup T . Suppose that $A \geq 3$ and denote $P_\alpha = T - M_\alpha$ and $M^* = \bigcap_\alpha M_\alpha$ we then have

- a) $P_\alpha \cap P_\beta \cap P_\gamma = \emptyset$ for $\alpha \neq \beta \neq \gamma$.
 b) $T = [\bigcup_{\alpha \in A} P_\alpha] \cup M^*$.
 c) For every $v \neq \alpha$ we have $P_\alpha \subset M_v$.
 d) If I is an ideal of T and $I \cap P_\alpha \neq \emptyset$ then $P_\alpha \subset I$.
 e) For $\alpha \neq \beta$ we have $P_\alpha \cap P_\beta \subset M^*$, so that M^* is not empty.

Proof: a) For $\alpha \neq \beta \neq \gamma$; we have $M_\alpha \cup M_\beta \cup M_\gamma = T$.

$$\text{Hence } P_\alpha \cap P_\beta \cap P_\gamma = (T - M_\alpha) \cap (T - M_\beta) \cap (T - M_\gamma) \\ = T - (M_\alpha \cup M_\beta \cup M_\gamma) = \emptyset.$$

- b) We have $M^* = \bigcap_{\alpha \in A} M_\alpha = \bigcap_{\alpha \in A} (T - P_\alpha) = T - \bigcup_{\alpha \in A} P_\alpha$.
 Hence $T = (\bigcup_{\alpha \in A} P_\alpha) \cup M^*$.

- c) For $v \neq \alpha$ we have $P_\alpha = T \cap P_\alpha = (M_v \cap P_\alpha) \cap P_\alpha = M_v \cap P_\alpha$. Hence $P_\alpha \subset M_v$.

- d) If $I \cap P_\alpha \neq \emptyset$ the set $M_\alpha \cup I$ is an ideal of T which is larger than M_α , hence $M_\alpha \cap I = T$.

Since $M_\alpha \cap P_\alpha = \emptyset$ we have $P_\alpha \subset I$.

- e) Suppose for an indirect proof that there is a couple $\mu_\alpha \in P_\alpha, \mu_\beta \in P_\beta$ such that $\mu_\alpha \mu_\beta = \mu_\gamma$ is not contained in M^* . With respect to (b) there is P_γ such that $\mu_\gamma \in P_\gamma$. Suppose first that $P_\gamma \neq P_\alpha$. Then $P_\alpha \subset T - P_\gamma = M_\gamma$ and $P_\alpha \cap P_\beta \subset M_\gamma \cap P_\beta \subset M_\gamma$, hence $\mu_\gamma \in M_\gamma$, which is a contradiction to $\mu_\gamma \in P_\gamma = T - M_\gamma$. Suppose next $P_\gamma = P_\alpha$. Then $P_\beta \subset T - P_\alpha = M_\alpha$ and $P_\alpha \cap P_\beta \subset P_\alpha \cap M_\alpha \subset M_\alpha$, hence $\mu_\gamma \in M_\alpha = T - M_\alpha$ which is contradiction to $\mu_\gamma \in P_\alpha$.

7. Theorem 4

Let T be a ternary semigroup containing maximal ideals and let M^* be the intersection of all maximal ideals of T . Then every prime ideal of T containing M^* and different from T is a maximal ideal of T .

Proof: Let Q be a prime ideal of T containing M^* and $Q \neq T$. We use the notations of Theorem 3. By (d) we have $Q = T - [\bigcup_{v \in H} P_v] = \bigcap_{v \in H} (T - P_v) = \bigcap_{v \in H} M_v$. Where

$H \subset A$ and H is not empty.

If $\text{card } H = 1$, we have $Q = M_v$, i.e. Q is a maximal ideal of T and our Theorem is proved.

We shall show that $\text{card } H \geq 2$ cannot take place. Suppose that for an indirect proof $\text{card } H \geq 2$. Let $\beta \in H$ and denote $M^1 = \bigcap_{v \in H, v \neq \beta} M_v$. We then have $Q = M^1 \cap M_\beta$ and

$M^1 M_\beta \subset M^1 \cap M_\beta = Q$. Since Q is prime, we have either $M^1 \subset Q$ or $M_\beta \subset Q$.

- 1) The first possibility $M^1 \subset Q$ together with $Q \subset M^1$ implies $Q = M^1$. Further $M^1 = Q = M^1 \cap M_\beta$ implies $M^1 \subset M_\beta$. By Theorem 3(c) we have $P_\beta = \bigcap_{v \in H} M_v = M^1$. Hence $P_\beta \subset M_\beta$ a contradiction with $P_\beta \cap M_\beta = \emptyset$.
- 2) The second possibility $M_\beta \subset Q$ together with $Q \subset M_\beta$ implies $Q = M_\beta$. Now $Q = M_\beta = M^1 \cap M_\beta$ would imply $M_\beta \subset M^1$. Since M_β is maximal and $M^1 \subset T$ we have $M^1 = M_\beta$. The relation $P_\beta \subset M^1 = M_\beta$ constitutes an apparent contradiction. This completes the proof of our Theorem.

Let now be $\mathfrak{R} = \{Q_\alpha / \alpha \in A_1\}$ the set of all prime ideals of T and different from T and $Q^* = \bigcap_{\alpha \in A_1} Q_\alpha$. Let $\mathcal{M} = \{M_\alpha / \alpha \in A\}$ be the set of all maximal ideals of T and (as above) $M^* = \bigcap_{\alpha \in A} M_\alpha$.

If T satisfies $T^3 = T$ and $\mathcal{M} \neq \emptyset$, $Q^* \neq \emptyset$, Theorem 1 implies $Q^* \subset M^*$.

8. Theorem 5

Let T be a ternary semigroup containing at least one maximal ideal. A prime-ideal $Q \neq T$ is a maximal ideal of T if and only if $M^* \subset Q$.

Proof: If Q is a maximal ideal, we clearly have $M^* \subset Q$. If conversely $M^* \subset Q$ then by Theorem 4, Q is a maximal ideal of T .

9. Theorem 6

Let T be a ternary semigroup containing at least one maximal ideal. Then every prime ideal of T (and $\neq T$) is a maximal ideal of T if and only if $M^* \subset Q^*$.

If more over $T = T^3$, we have $Q^* \subset M^*$.

10. Theorem 7

Let T be a ternary semigroup with $T = T^3$, containing at least one maximal ideal. Then $\mathfrak{R} = \mathcal{M}$ if and only if $Q^* = M^*$.

11. References

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