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Verification on a Given Point Set for a Cubic Plane Graph

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Abstract

Cubic graph, where all vertices have degree three can be associated as 3-regular graph and trivalent graph. Let a point P be the given point set, with a stipulation of $n \geq 4$ points in the regular plane such that n is even in general situation. It has been probed that the problem whether there is a straight-line cubic plane graph on *P*. Algorithm is not known for polynomial-time of a problem. With research on the reduction to the existence of the convex hull of *P* along with the certain diagonals of the boundary cycle, the polynomial-time algorithm is proposed that checks for 2-connected cubic plane graphs. The algorithm is constructive and runs in time complexity of $O(n3)$. It is also shown that which graph structure can be expected when there is a cubic plane graph on *P*; e. g., a cubic plane graph on *P* implies a connected cubic plane graph on *P*, and a 2-connected cubic plane graph on *P* implies a 2-connected cubic plane graph on *P* that contains the boundary cycle of *P*.

Keywords: Algorithm, Cubic Graph, Euler's Formula Plane Graph, Polynomial Time Algorithm, Trivalent Graph

1. Introduction

Let P be a set of $n \geq 4$ points in the plane that is in general position, i.e., that does not contain three points on a line. A straight-line embedding of a graph G = (V, E) is an injective function $\pi: V \rightarrow R^2$ such that for any two distinct edges ab and cd the straight line segments $\pi(a)\pi(b)$ and $\pi(c)\pi(d)$ are internally disjoint (i.e., they may only intersect at their endpoints). Let P admit a graph G = (V, E) if $|P| = |V|$ and there is a straight-line embedding that maps V to P; we also have G is on P and P can admit only plane graphs.

We are interested in classifying the point sets P that admit at least one simple plane graph G with a given additional property, e.g., being k-connected, k-edge-connected or k-regular. The graph G is not part of the input: it suffices to find any graph G on P with the desired properties[8]. Using Euler's formula, none of these properties can exist for $k \geq 6$, so we focus on $k \in \{0, 5\}$. In addition, there are no k-regular graphs for k = 1, 3, 5 when n is odd, since every graph must have an even number of odd vertices. Let us assume in these cases that n is even. Since we are dealing only with simple graphs, we can further assume $n > k$ throughout the paper. If not stated otherwise, all graphs are assumed to be simple and plane, but not necessarily connected. For $k \in \{0, 1\}$, it is easy to see that every point set admits a 0-connected 0-regular as well as a connected graph, but 1-regular graph can only be found when n is even (Table 1). For k = 2, every point set P admits a 2-regular 2-connected (and thus also 2-edge-connected) graph, as there is always a plane cycle on P[12].

For k = 3, Dey et al.[23] give a construction proving that there is a 3-connected graph on P if and only if $n > 3$ and P is not in convex position (the same characterization holds for 3-edge- connected graphs). Garcia et al.[1] investigated of how many edges are sufficient to allow a 3- connected graph on P. Consider h be the number of points on the convex hull boundary of P. If P is not in convex position, it gives the construction of a 3-connected graph on P that has max ($\frac{3}{2}$ n, n + h − 1) edges; the same construction

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Table 1. Conditions on P that are both necessary and sufficient for the existence of a k-connected, k-edge-connected and k-regular plane graph, respectively, on a point set P in general position, where $|P| = n > k$

	Necessary and sufficient conditions for a				Minimum number of edges in a k-(edge) connected Plane graph on P
k	k-connected plane graph	k-edge-conn ected plane graph	k-regular plane graph		plane graph on P
0	None	None	none		0
1	None	None	n even		n – 1
2	None	none	none		N
3	P not in convex position	P not in convex position	? (known for h ¾ n)	≤	max(¾ n, n + h-1)
4	? (known for h = 3)	?	?		?
5	?	?	?		?

on minimalist constraints holds for 3-edge-connected graphs. They also prove that this number is minimal for any 3-connected and for any 3-edge-connected graph on P[10].

As a corollary, $h \leq \frac{n}{2} + 1$ implies the existence of a 3-regular graph on P. Garcia et al. show in addition that there is a 3-regular graph on P (not necessarily 3-connected, but still 2-connected) when $\frac{n}{2} + 1 \leq h \leq \frac{3}{4} n^3$, Theorem[4]. While this gives a characterization of the point sets admitting 3-regular graphs for $h \leq \frac{3}{4}$ n, the problem remained open for higher values of h. Examples show that the existence of a 3-regular graph is then not any more dependent only on h and n[1]. We give a characterization for all values of h, which leads to the first polynomial time algorithm that computes a 3-regular graph on P if it exists; the running time is O (n^3)[11]. We also show that the existence of a 2-connected cubic graph on P implies that there is also a 2-connected cubic graph on P that contains the boundary cycle of the convex hull of P.

If we do not insist on having degree 3 for every vertex, the result by Gritzmann et al.[20] that every outer planar graph is on P can be applied: We can always find a 2-connected outer planar graph on P that has all except two or three vertices of degree 3. In contrast to $k \leq 3$, very little is known about the case $k \in \{4, 5\}$. There exist point sets that are neither in convex position nor admit 4-connected graphs. For the special case of h = 3, Dey et al.[23] could characterize the point sets that admit 4-connected graphs.

1.1 Preliminaries

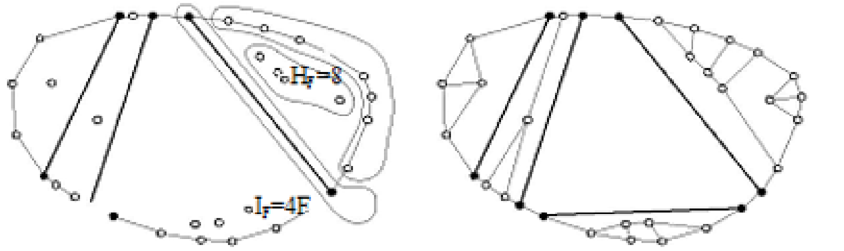
A graph is cubic if it is 3-regular. Let P be a set of $n \geq 4$ points in the plane in general position with n even. Let ch (P) denote the boundary cycle of the convex hull of P and let a (combinatorial) edge in be a diagonal of P if it joins two non-consecutive points in ch (P). Let H be the set of points in ch (P), h: = |H|, and let I = P \ H be the set of inner points in P, i = |I|.

Let D be a set of non-crossing diagonals of P. let's call the bounded regions in which ch (P) and D subdivides the plane faces induced by D[13]; let F (D) be the set of faces induced by D. For every induced face $f \in F(D)$, let V_f be the set of endpoints of diagonals on the boundary off, let H_f be the set of points on the boundary off that are not in V_f and let I_f be the set of points strictly inside f (see Figure 1). Moreover, let $i_f := |I_f|$, $h_f := |H_f|$, $v_f := |V_f|$, $N_f := H_f \in I_f$ and $n_f = |N_f|$.

For a vertex $v \in V(G)$ and $X \in V(G)$ of a graph G, let G[X] be the sub graph of G that is vertex-induced by X. For a simple cubic plane graph G on P and a face $f \in F(D)$, assume $G[f] = G[N_f \in V_f]$.

2. Necessary Conditions for Cubic Graphs

Every simple cubic plane graph G on P contains a (possibly empty) set D of non-crossing diagonals on P. Such a way, G naturally defines a set of induced faces. For any non-diagonal edge, the induced face contains e, it is only important how many non-diagonal edges in an induced face f are incident to a vertex $p \in P$[14]. The precise neighbors of p in f are not crucial. This is modeled by representing non-diagonal edges with half-edges, i. e., we specify for every vertex the number of incident half-edges, but not the neighbors of that vertex.



(a) (b)

Figure 1. Induced faces. (a) The faces induced by a set of non-crossing. (b) A cubic graph on diagonals. Each diagonal is drawn with a thick edge point set of Figure (a) and black end vertices using the given diagonals.

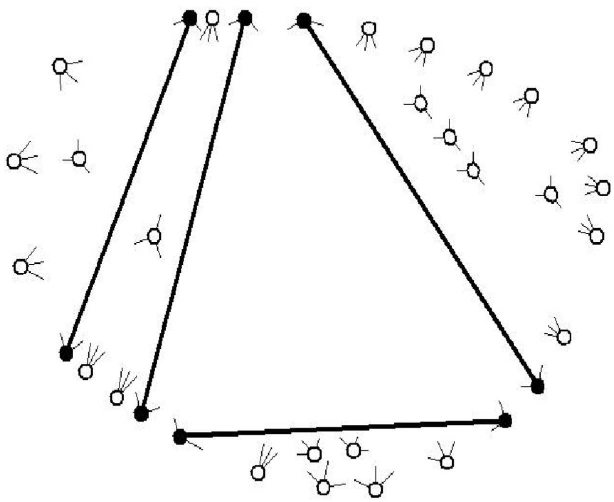


Figure 2. The diagonal configuration of the cubic plane graph.

- Every half-edge on p is assigned to a face that is induced by D(C) and contains p.

Every simple cubic plane graph G on P determines a unique diagonal configuration C_G by cutting every non-diagonal edge e into two half-edges such that both half-edges are assigned to the induced face that contains e; see Figure 2 for an example. We list necessary conditions on graphs and diagonal configurations to allow cubic graphs on P. Garcia et al. has proved the following

Theorem 1. ([1], implicitly)**.** Let P be a set of $n \geq 4$ points in general position such that n is even. If $h \leq \frac{3}{4}$ n, there is an algorithm with running time O (n^3) that constructs a simple cubic 2-connected plane graph on P that contains ch (P). If h = n, P does not admit a simple cubic plane graph[16].

We can therefore focus on the case $h > \frac{3}{4}$ n; then, every cubic plane graph on P must contain at least one diagonal.

Lemma 1. Let $h > \frac{3}{4}$ n. For every simple cubic (not necessarily connected) plane graph G on P, D (C_G) 6= ∅. Moreover, $|D\ (C_G)| \geq 2(h - \frac{3}{4}\ n) = \frac{h-3i}{2}$.

Proof: Every vertex v in H \ V (D (C_G)) has at least one neighbor in I, as we can have at most two neighbors in H and has degree 3. Consider s be the number of edges in G that join vertices of H with vertices of I. Then $h - 2|D\ (C_G)| \leq s$. However, $s \leq 3i$, because every inner vertex can be adjacent to at most 3 vertices in H. It follows that $|D\ (C_G)| \geq \frac{h-3i}{2} = 2(h - \frac{3}{4}\ n)$. If $h > \frac{3}{4}$ n, then the right-hand side will be positive, which proves the theorem[18].

To establish further conditions on diagonal configurations for cubic graphs, we classify the vertices in each induced face f.

A vertex in V_f that is not balanced is also called unbalanced. Let V_f^0, V_f^+ and V_f^- be the set of balanced, hungry and sated vertices in V_f, respectively. For a sated vertex $v \in V_f^-$, let w be the unique neighbor of v in ch (P) that is contained in f. If w is also sated in f, v is called matched (with w) and {v, w} is called a matched vertex pair. Otherwise, v is unmatched. Let V_f^{-m} and V_f^{-u} be the set of matched and unmatched vertices in V_f, respectively. Let $v_f^+ = |V_f^+|$, $v_f^- = |V_f^-|$ and $v_f^{-u} = |V_f^{-u}|$. We now assign each induced face f the following integer value Δ(f) in order to prove necessary conditions on these values for cubic graphs.

Definition: For a diagonal configuration C on P and each induced face $f \in F$ (D(C)), let $\Delta\ (f) = 3i_f - h_f - v_f^+ - v_f^{-u}$. It is useful to imagine $h_f + v_f^+ + v_f^{-u}$ as the number of edges between inner and boundary vertices off that are necessary for any cubic graph on P. Note that an unmatched vertex requires such an edge indirectly [21]. It forces its non-sated neighbor to require one additional such edge.

Lemma 2. Let G be a simple cubic plane graph on P**.** For every face $f \in F$ (D (C_G)), $\Delta(f) \geq 0$.

Proof: Let f be a face in F $(D(C_G))$. We count the number s_f of edges in G that are incident with exactly one vertex in I_f. Each of the v_f^+ hungry vertices in V_f is incident to at least one such edge. There is no diagonal of $H_f \cup V_f$ in G, as otherwise this diagonal would be a diagonal of P due to $H_f \cup V_f \subseteq H$ and contradict f to be an induced face. Therefore, every vertex v in H_f has at most two neighbors in $H_f \cup V_f$ and, hence, at least one neighbor in I_f. Each unmatched vertex $v \in V_f$ has a unique neighbor w in ch(P) $\cap H_f$ such that $vw \in/E(G)$. This forces w to be incident with an additional edge to a vertex in I_f for each such vertex v. It follows that $h_f + v_f^+ + v_f^{-u} \leq s_f$. Since $s_f \leq 3i_f$, $\Delta(f) \geq 0$.

Let a set of diagonals be disjoint if no two of them contain a common end vertex. Note that D (C_G) in Lemma 6 does not have to be disjoint [22]. However, if G contains ch (P), all diagonals are disjoint and consist of balanced vertices only, which give the following corollary from Lemma 1.

Corollary 1. Let G be a simple cubic plane graph on P that contains ch (P). Then $h_f \leq \frac{3}{4} n_f$ for every induced face $f \in F$ (D (C_G)).We show a necessary parity condition for $\Delta(f)$.

Lemma 3. Let G be a simple cubic plane graph on P. Then $\Delta(f)$ is even for every induced face $f \in F$ $(D(C_G))$.

Proof: Consider the graph G[f] and its vertex set $I_f \cup H_f \cup V_f^+ \cup V_f^{-u} \cup V_f^{-m} \cup V_f^0$. By definition, the degree in G[f] of all vertices in V_f^0 is even while the degree of all other vertices in G[f] is odd. As every graph has an even number of odd-degree vertices, $i_f + h_f + v_f^+ + v_f^{-u} + v_f^{-m}$ must be even. However, v_f^{-m} is even, as matched vertices come in pairs. Thus, $i_f + h_f + v_f^+ + v_f^{-u}$ is even and it follows that $3i_f - h_f - v_f^+ - v_f^{-u} = \Delta$ (f) is even.

3. Constructions

We give sufficient conditions for diagonal configurations to admit cubic graphs. The following result of Tamura and Tamura will be used.

Lemma 4 (Tamura and Tamura[2]). For points $p_1, p_2, \ldots, p_n$ in general position in the plane and any assignment of degrees from $\{1, 2, \ldots, n - 1\}$ to the points such that the sum of degrees is $2n - 2$, there is a plane tree with these prescribed degrees on $p_1, p_2, \ldots, p_n$. Moreover, the plane tree can be constructed in time O (n log n)[1,16]. We call an induced face f empty if $i_f = h_f = 0$. The following lemmas clarify for which diagonal configurations we can expect cubic plane graphs and, for a special type of configurations, how these graphs can be constructed.

Lemma 5. Let C be a diagonal configuration without unmatched vertices such that for every non-empty induced face $f \in F$ (D(C)), $\Delta(f)$ is even and $0 \leq \Delta(f) < 2i_f$. Then there is a simple cubic plane graph G on P with D (C_G) = D(C). If C has additionally no matched vertices, there is a simple cubic 2-connected plane graph G on P with D (C_G) = D(C) that contains the boundary cycle of P.

Proof: The proof for the first claim builds on a construction given in[1], but avoids the creation of additional diagonals. Let G^0 be the graph that consists of D(C) and of all edges in ch (P) that do not join a matched vertex pair. A cubic graph is constructed by adding edges to G^0 in each induced face $f \in F$ (D(C)). As there is no unmatched vertex in D(C), every vertex in $H_f \cup V_f^+$ of an induced face f needs exactly one additional incident edge to the interior off. Note that the inner vertices if in f are not necessarily contained in the convex hull of $H_f \cup V_f^+$.

In each induced face $f \in F$ (D(C)), we augment G^0 by a collection L of plane trees, with at least three vertices, that

1. The union of all trees is plane,
2. Every vertex v in $I_f \cup H_f \cup V_f^+$ is contained in exactly one tree $T \in L$ and
3. V has degree 3 in T if $v \in I_f$ and degree 1 in T if $v \in H_f \cup V_f^+$.

In particular, no tree in L contains a boundary edge off, as it contains at least three vertices. We prove that the second claim can be deduced from the first. As there are neither matched nor unmatched vertices, C has only balanced vertices. Thus, G^0 = ch (P) $\cup$ D(C) is a cycle with chords. The constructed graph contains two internally vertex-disjoint paths from every vertex in I_f to distinct boundary vertices off [19]. Therefore, the constructed graph must be 2-connected if D(C) is disjoint.

Let f is an induced face. According to Lemma 3, Δ (f) $\leq 2i_f - 2$. Let k be the integer such that Δ (f) $= 2i_f - 2k$, i. e., $1 \leq k \leq i_f$. We first show how to construct a collection of k plane trees $T_1, T_2, \ldots, T_k$ that satisfy the properties (2) and

(3). The first k – 1 trees are chosen as any collection of vertex-disjoint trees $K_{1,3}$ such that the degree-3 vertex of every $K_{1,3}$ is in I_f.

For convenience, let $z = h_f + v_f^+$. Recall that Δ (f) = $3i_f - z$ and that z is the number of distinct boundary vertices off that need exactly one additional incident edge to some inner vertex in f[24]. Choosing the first k – 1 trees as described above leaves precisely $i_f - k + 1$ inner vertices of f and $z - 3k + 3$ vertices in $H_f \cup V_f^+$, giving a total of $i_f + z - 4k + 4$ vertices. Thus, the degrees that are still needed for these remaining vertices sum up to $3i_f - 3k + 3 + z - 3k + 3 = 2i_f + 2z - 8k + 6$, which is equal to $2(i_f + z - 4k + 4) - 2$. Therefore, we can apply Lemma 2 to construct a last plane tree T_k with properties (2) and (3).

Now choose L as a collection of plane trees on at least three vertices that satisfies (2) and (3) and for which the sum of all edge lengths is minimal. This collection exists, as we know that $T_1, \ldots, T_k$ exists. We need to show that L satisfies (1). Assume to the contrary that two edges ab and cd that are contained in distinct trees, cross. Using the triangle inequality,

We can delete ab and cd and either add the edges ac and bd or the edges ad and bc to generate two collections L1 and L2 of trees that have a smaller sum of edge lengths. As at most two vertices of {a, b, c, d} can be boundary vertices off, at most one of L1 and L2 contains a boundary edge off. The other collection then preserves properties (2) and (3), which gives a contradiction to L being minimal; thus, L satisfies (1).

Figure 3: An induced face f with $\Delta(f) = 2i_f = 6$ that does not admit a simple cubic plane graph. Delete ab and cd and either add the edges ac and bd or the edges ad and bc to generate two collections L1 and L2 of trees that have a smaller sum of edge lengths. As at most two vertices of {a, b, c, d} can be boundary vertices off, at most one of L1 and L2 contains a boundary edge off. The other collection then preserves properties (2) and (3), which gives a contradiction to L being minimal; thus, L satisfies (1).

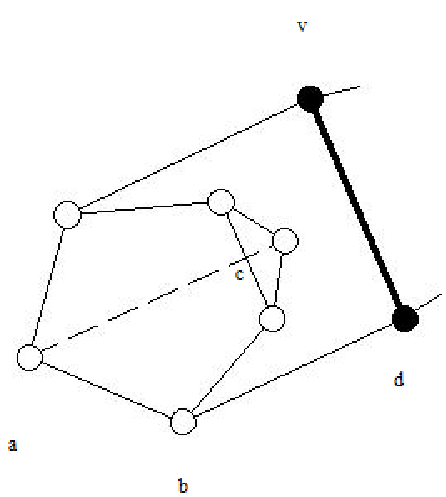


Figure 3. An induced face f with Δ (f) = $2i_f$ = 6 that does not admit a simple cubic plane graph.

Remark: Intuitively, the precondition $\Delta(f) < 2i_f$ in Lemma 4 avoids that we have to build cycles in the induced face f. I_f Δ (f) $\geq 2i_f$, the desired sum of degrees for L would exceed the sum of degrees of every forest in f. The precondition is tight for the statement of Lemma 4, as there are counterexamples even for $\Delta(f) = 2i_f$ ($\Leftrightarrow i_f = h_f + v_f +$), each of the vertices a, b and c has to be adjacent with exactly one of the 3 white boundary vertices for a cubic graph in face f. So, a, b and c should form a triangle, which has edge from b to the boundary to induce a crossing.

Lemma 6. Let C be a diagonal configuration without unmatched vertices and Δ (f) = 0 for every induced face f $\in$ F (D(C)). There is a O(n log n) algorithm that constructs a simple cubic plane graph G on P with D(CG) = D(C) and no edge that joins two points of I.

Proof: We create a graph G0 and, for every induced face f $\in$ F (D(C)), a collection L of trees satisfying properties (1)–(3), as shown in the proof of Lemma 10. As Δ(f) = 0, the number of vertices in $H_f \cup V_f+$ for every face f is exactly 3if and every such vertex needs exactly one additional incident edge to the interior off. Thus, L must consist of if trees, each of which is a $K1^3$. We take two point sets of equal size: one is $Z = H_f \cup V_f+$ and the other set Y is generated from If by replacing each point p $\in$ If with three points that are in a sufficiently small -neighborhood of p (such an can be found efficiently).

We need to compute a plane matching between Y and Z. Clearly, it can be assumed that $Y \cup Z$ is in usual position. A ham-sandwich cut for Y and Z to be computed in time O (n)[7]. Iterating the computation for the subsets of $Y \cup Z$ that are contained in each side of the cut, respectively, will terminate with cells containing exactly one point of Y and one point of Z. Joining the two points for every cell by an edge constructs L, giving a total running time of O(n log n).

4. Reduction Education to Diagonal Configurations

We have shown that a suitable diagonal configuration allows constructing a cubic graph on P. If we can show that every cubic graph on P implies the existence of such a suitable diagonal configuration, the problem of computing cubic graphs to diagonal configurations is reduced[15]. We will give an efficient algorithm for finding such a diagonal configuration in the next section. Let h > 34 n due to Theorem 2.

Let CG be the diagonal configuration of a simple cubic plane graph G on P. CG is to be transformed to the desired diagonal configuration by iteratively applying four operations. In each step, we maintain a diagonal configuration C. We emphasize that the operations are applied in the order of appearance, i.e. an operation is only applied if the operations before are not applicable.

4.1 Operation

Let Δ (f) > 0 for an induced face f of D(C) (Figure 3). Cut a diagonal vw on the boundary off into two (non-diagonal) half-edges. Both half-edges are assigned to the newly generated induced face z.

4.2 Operation

Let v ∈ Vf–u for an induced face f of D(C). Cut the unique diagonal vw on the boundary off into two (non-diagonal) half-edges. Both half-edges are assigned to the newly generated induced face z.

4.3 Operation

Let f be an induced face of D(C) whose boundary vertices consist of exactly two matched vertex pairs {v, w} and {x, y}. Cut the diagonals vy and wx into two (non-diagonal) half-edges. All the half-edges are allocated to the newly generated induced face z.

4.4 Operation

Let none of the Operations 1–3 be applicable and let f be an induced face of D(C) that contains two matched vertex pairs {v, w} and {x, y} in this order counter clockwise[17]. We denote the unique neighbors of v and won the boundary off by v^0 and w^0, respectively. Note that it is possible that v^0 = y or w^0 = x but not both. Consider g1 and g2 are faces that are separated from f by vv0 and ww0, respectively.

4.4.1 Operation

The quadrangle {v, v_0, w_0, w} contains no point in If. Cut the diagonals vv0 and ww0 into two (non-diagonal) half-edges each and add the diagonal v0w 0. Let z and z0 be the two new induced faces separated by v0w 0 such that z0 contains v. The two new half-edges at v and w are assigned to z0 and all half-edges that were originally in f are assigned to z.

4.4.2 Operation

The quadrangle {v, v_0, w_0, w} contains a point in if,

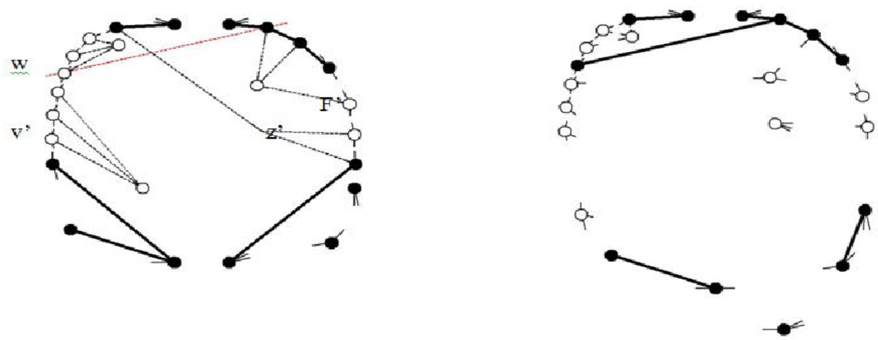
We partition V_f+ ∪ H_f into the two subsets X^0 and Y^0 such that X^0 contains the points of V_f+ ∪H_f to the left of the line vy (oriented from v to y) and Y^0 contains the remaining points of V_f+ ∪ H_f. Note that $X^0 \cup Y^0$ does not contain any of the vertices {v, w, x, y} and that no unmatched vertex exists due to Operation 2; hence, Δ(f) is only dependent on if , $|X^0|$ and $|Y^0|$. We set X = $X^0 \cup \{v^0\}$ and Y = Y 0 ∪ $\{w^0\}$. Let $x^1, x^2, \ldots, x^s$ with x1 = v^0 be the points in X ordered clockwise and let $y_1, y_2, \ldots, y_t$ with y_1 = w0 be the points in Y ordered counterclockwise.

For i ∈ {1, . . . , s} and j ∈ {1, . . . , t}, let Cij be the diagonal configuration obtained from C by removing the diagonals vv0 and ww^0 and adding the diagonal x_iy_j. Let z and z^0 be the two new induced faces separated by x_iy_j such that z^0 contains v. We will replace C by some C_{ij} such that Δ(z) = 0. The two new half-edges at v and w are assigned to z^0. It remains to assign the half-edges at v^0, w^0, xi and yj that were originally in f and that are not used for creating the new diagonal xiyj: The half-edges at v^0 and w^0 are assigned to z^0. If xi 6 = v^0, the half-edges at xi are distributed to z and z^0 in the unique way such that xi is neither sated in z nor in z^0; the half-edges at y_j when y_j 6= w^0 are assigned accordingly. The Operation 4.4.2 does not create any new sated vertex in z and z^0 due to this assignment. Note that every of the Operations 1–4 decrease the number of diagonals by at least one.

Lemma 7. In Operation 4.2, i ∈ {1, . . . , s} and j ∈ {1, . . . , t} can be chosen such that Δ(z) = 0 in C_{ij} and such that i = 1 or j = t.

Proof: Since Operation 1 is not applicable, Δ(f) = 0 before Operation 4.2. If i = j = 1, Δ(z) < 0 in C_{ij}, since Δ(f) = 0 and the quadrilateral vv^0w^0w contains at least one point of I_f . If i = s and j = t, Δ(z) ≥ 0 in C_{ij}. Changing C_{ij} to $C_{(i+1)j}$ either increases the value of Δ(z) by one (if the wedge between the two rays x_iy_j and $x_{i+1}y_j$ contains no point of I_f) or decreases Δ(z). Similarly, if C_{ij} is changed to $C_{i(j+1)}$, the value of Δ(z) either increases by one or decreases. It follows that Δ(z) = 0 for at least one of the diagonal configurations $C_{11}, C_{12}, \ldots, C_{1t}, C_{2t}, \ldots, C_{st}$.

For being able to construct a cubic graph from a diagonal configuration, we need in particular that Δ(f) ≥ 0 and Δ(f) is even for every induced face f. We prove that the above operations preserve these properties of C in every step.



(a) (b)

Figure 4. (a) Before Operation 4.4.1. The dotted edges depict which deletes the diagonals. (b) After Operation 4.4.2, possible cubic graph.vv0 and ww0 and inserts the diagonal x5y5.

Table 2. The five possible configurations of a diagonal end vertex v before Operation 1

v ϵ	Δ(f)	v ϵ	Δ(g)	v ϵ	Δ(z)	effect on Δ(z)
V_f^0	+0	V_g^0	+0	$H_z \cup V_z^+$	−1	-1
V-u		V^0		V^0		
F	-1	G	+0	Z	+0	+1
V-u		V^+		H V^+		
F	-1	G	-1	z ∪ Z	-1	+1
V-m		$_V 0$		$_V 0$	(?)	
F	+0	G	+0	Z	-1	-1
V - m		$_V +$		H $_V$ +	(?)	
F						
	+0	G	-1	Z U Z	-2	-1

Lemma 8. Operations 1–4 preserve for every induced face f^0 that $\Delta(f^0) \geq 0$ and that $\Delta(f^0)$ is even.

Proof: Consider the diagonal configuration C before an Operation 1. As D(C) 6 = ∅ due to h > $\frac{3}{4}$ n, at least one diagonal vw is present. Consider g is the face of D(C) which is separated from f by vw. According to Lemmas 5 and 6, $\Delta(f) \geq 2$ and $\Delta(g) \geq 0$. We check the effects of Operation 1 on Δ(z) in dependence of the type of v. By symmetry, the same effects hold for w. This lists the five possible configurations for vertex types of v in f and g. We illustrate the most involved case $v \in V_f^{-m}$ and $v \in V_g^{+}$.

Since $v \in V_f^{-m} \cap V_g^{+}$, the contribution of v to Δ(g) in C is −1, while the contribution of v to Δ(f) is 0, as matched vertices do not influence . Applying Operation 1 replaces these two contributions to Δ(f) and Δ(g) with one contribution to Δ(z). The difference between the new and the two old contributions of v is called the effect of v on Δ(z); it reflects the difference of the -values before and after the operation in dependence of v. If v has been incident to only one diagonal of D, v will be a new vertex in H_z (and, thus, in N_z) after performing Operation 1, causing Δ(z) to decrease by 1. Otherwise, v will be in V_z^+, which also decreases Δ(z) by 1. Additionally, Δ(z) decreases by another 1 in either cases, as the vertex in f that was formerly matched to v is now unmatched. In total, the effect of v on Δ(z) is thus −2 + 1 = −1. Table 2. lists the effects for the other four cases.

Columns 2 and 4 depict the contribution of v to the given value. Column 6 depicts the contribution of v to Δ(z) after Operation 1. For entries marked with, the sated vertex in f that was matched to v becomes unmatched in z.

According to Table 2, Δ(z) = Δ(f)+Δ(g)+x with x ∈ {−2, 0, 2} after applying Operation 1, since exactly two vertices change. This implies $\Delta(z) \geq 0$ and that Δ(z) is still even[24].

Consider the diagonal configuration C before an Operation 2. As $v \in V_f^{-u}$, v must be either contained in V_g^{0} or in V_g^{+}. In both cases, the effect of v on $\Delta(z)$.Consider the diagonal configuration C before an Operation 3. All four vertices v, w, x and y are matched and therefore either in $V_{g\,i}^{0}$ or in $V_{g\,i}^{+}$ for $i \in \{0, 1\}$. Cutting the diagonal vy therefore gives twice an effect of −1 on the new induced face, according to Table 2. Cutting the diagonal wx then gives twice an effect of +1 on $\Delta(z)$. Thus, after applying Operation 3, $\Delta(z) = \Delta(f) + \Delta(g_1) + \Delta(g_2)$ and the claim follows.

Consider the diagonal configuration C before an Operation 4.4.1. Note that applying Operation 4.1 yields $\Delta(z) = \Delta(f) = 0$, as v and w do not contribute anything to $\Delta(f)$. Note that $\Delta(z^0)$ differs from $\Delta(f) + \Delta(g_1) + \Delta(g_2) = 0$ only by the effects of the vertices v and w, as $\Delta(z) = 0$. Similarly as for Operation 3, the effects of v and w on $\Delta(z^0)$ cancel each other. Thus, $\Delta(z^0) = 0$, which gives the claim.

Consider the diagonal configuration C before an Operation 4.4.2. Then $\Delta(f) = \Delta(g_1) = \Delta(g_2) = 0$. Since $\Delta(z) = 0$ in C_{ij}, it remains to show that $\Delta(z^0)$ is even and non-negative. Note that $\Delta(z^0)$ differs from $\Delta(f) + \Delta(g_1) + \Delta(g_2) = 0$ only by the effects of the vertices therefore balanced in z and z^0, as they were formerly contained in $H_f \cup V_f$. Thus, the effect of x_i and y_j on $\Delta(z^0)$ is +1 each. The effect of the vertices v and w cancels each other, as shown for Operation 3. The effect of v^0 if v^0 6= x_i and of w^0 if w^0 6= y_j is given by Table 2: Since no vertex is unmatched in C, the effect is −1 per vertex. This amounts to a total effect of +0, which gives $\Delta(z^0) = 0$ and the claim.

Theorem 2. The following statements are equivalent:

1. P admits a simple cubic plane graph G.
2. P admits a simple connected cubic plane graph G^0.
3. $h \leq \frac{3}{4}$ n or there is a diagonal configuration C on P such that $\Delta(f) = 0$ for every induced face $f \in F(D(C))$ and no vertex is unmatched.

Proof: The proof for (2) ⇒ (1) is immediate. We prove (1) ⇒ (3). Assume that $h > \frac{3}{4}$ n. Then $D(C_G)$ 6= ∅ by Lemma 3 and we can iteratively apply (any of the) Operations 1 and 2 on C_G as long as possible until the process terminates with a diagonal configuration C^0. Note that the application of each operation decreases the number of diagonals by one; thus, $D(C^0) \subseteq D(C_G)$. Due to Lemma 7, every induced face $f \in F(D(C^0))$ satisfies $\Delta(f) = 0$ and no vertex can be unmatched.

We prove (3) ⇒ (2). If $h \leq \frac{3}{4}$ n, the proof follows directly from Theorem 2. Assume that $h > \frac{3}{4}$ n. Let C^0 be the diagonal configuration obtained from C by applying Operations 1–4 to C as long as possible. Every face f induced by the diagonals of C^0 satisfies $\Delta(f) = 0$ and contains neither an unmatched vertex nor more than one matched vertex pair. Applying Lemma 11 to C^0 therefore constructs a cubic plane graph on P that is connected.

Corollary 2. If $h > \frac{3}{4}$ n, every simple cubic plane graph on P implies the existence of a simple connected cubic plane graph G on P that contains no unmatched vertex. By joining two vertices in I, every induced face $f \in F(D(C_G))$ and having $\Delta(f) = 0$ for every such face f.

Theorem 3. The following statements are equivalent:

1. P admits a simple cubic 2-connected plane graph G^0 such that G^0 contains ch(P), C_G0 has no unbalanced vertex and the diagonals $D(C_G0)$ are disjoint.
2. $h \leq \frac{3}{4}$ n or there is a diagonal configuration C on P such that there are no unbalanced vertices and $h_f = \frac{3}{4} n_f$ for each induced face $f \in F(D(C))$.

Proof: The proof for (2) ⇒ (1) is immediate [21]. We prove (3) ⇒ (2). In case that $h \leq \frac{3}{4}$ n, Theorem 2 settles the claim. Let $h > \frac{3}{4}$ n. Then D(C) must be disjoint, as every vertex v that is end point of two diagonals would contradict that v is balanced in every induced face. With $h_f = \frac{3}{4} n_f$ and $v_f^+ = v_f^- = 0$ for every induced face f, $\Delta(f) = 0$ follows. Applying the construction of Lemma 11 yields the desired graph G^0 and ensures ch(P) ⊆ G^0

It remains to prove (1) ⇒ (3). We can assume $h > \frac{3}{4}$ n. Then G consists of at least one diagonal by Lemma 3. The unbalanced vertex in C_G would imply G to contain a cut vertex. Hence there are no unbalanced vertices in C_G. Applying Operation 1 on C_G results in a diagonal configuration that satisfies $\Delta(f) = 0$ for every induced face f. As Operation 1 does not introduce new unbalanced vertices, $h_f = \frac{3}{4} n_f$ for every f, which gives the claim.

The properties that can be deduced from not being able to apply Operations 1–4 are valid also for cubic plane graphs that contain the least possible number of diagonals for P.

Corollary 3. If $h > \frac{3}{4}$ n, every simple 2-connected cubic plane graph on P that contains the least possible number of diagonals contains ch (P) and no unmatched vertex.

In particular, the number of diagonals in these graphs is completely determined by n and h: Given n and h, the number of diagonals is $2h - \frac{3}{2}$ n, which follows from Lemma 3 and the construction of Lemma 9, in which no two vertices in I are joined by an edge. Additionally, h > $\frac{3}{4}$ n implies for every induced face f that every vertex in I_f can be joined to exactly three vertices in H_f . We get the following corollary.

Corollary 4. If h $\frac{3}{4}$ n, every simple cubic 2-connected plane graph on P implies the existence of a simple cubic 2-connected plane graph G on P such that ch (P) $\subseteq$ G, $D(C_G)$ is disjoint, $|D(C_G)| = 2h - \frac{3}{2}$ n and there is no edge in G that joins two vertices in I[24]. One could be tempted to prove that every point set P admitting a connected cubic plane graph also admits a 2-connected cubic plane graph. But, this is not true with of the following example.

Lemma 9. There is no simple 2-connected cubic plane graph on the point set P.

Proof: Assume to the contrary there is such a graph G. Since h >– $\frac{3}{2}$ n, we may assume with Corollary that G contains ch (P) and exactly 3 disjoint diagonals. For every diagonal Z with an end vertex in {x, y, z, k, l, m}, let hp (Z) be the open half plane defined by Z. Note that hp (Z) does not also contain b. Assuming that there is a diagonal ending at a vertex in {x, y, z, k, l, m}, we choose one such diagonal Z with a minimal number of points in

Hp (Z). As hp (Z) is non-empty, but contains no inner vertex, every vertex in hp (Z) has degree 2, contradicting the chubbiness of G. This leaves the six remaining candidates {1, . . . , 6} for an end vertex of a diagonal. No diagonal can join two vertices of {1, . . . , 4}, as these vertices form a path in ch (P). Thus, there can be only two disjoint diagonals that have end vertices 5 and 6, respectively, contradicting that there have to be 3 disjoint diagonals.

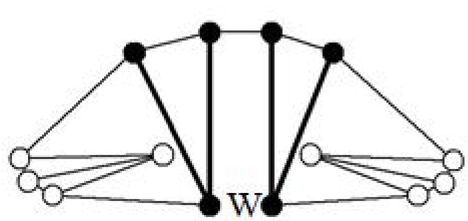


Figure 5. A point set that admits no 2-connected cubic plane graph but a connected cubic plane graph.

5. The Algorithms

In this section we describe the algorithms. An algorithm for finding a 2-connected cubic plane graph on a given point set P (if it exists)[17]. By Theorem 2, it suffices to look for a plane graph G that contains all edges of ch (P) and in which all faces induced by the set of the diagonals of G satisfy $\Delta(f) = 0$. Then we give a similar, very technical algorithm for finding a cubic plane graph (not necessarily connected) on a given point set P (if it exists). In both cases we use a dynamic programming approach. For an ordered pair (a, b) of points, let R (a, b) be the closed half plane to the right of the line ab (oriented from a to b). For a point $x \in H$, let x^+ and x^- be the points of H counterclockwise and clockwise of x.

Observation 1. Let $(a, b) \in T$ and let D be defined as. We distinguish the following two cases.

(i) D does not contain the diagonal ab.

Let f be the unique face induced by D that contains the segment ab and some points to the right of the segment ab. Then there is a point $c \in H(a, b) \cup \{a\}$ such that the face f contains the pair c^-, c^+ of adjacent vertices of ch(P). Consequently, every diagonal of D lies entirely in one of the closed half spaces R(a, c) and R(c^+, b).

(ii) D contains the diagonal ab, all the other diagonals of D lie entirely in R(a, b).

5.1 Algorithm for Finding a 2-Connected Cubic Graph

If n is odd or $n \leq 3$, there is no cubic graph on P ; we therefore assume that n is even and $n \geq 4$. If $h \leq \frac{3}{4}$ n, we can find a 2-connected cubic plane graph on P in time O (n^3) due to Theorem 2. Let h > $\frac{3}{4}$ n. In particular, h > 3.

For all pairs $(a, b) \in T$ with $|H(a, b)| \leq 2$, no diagonal in H(a, b) can exist and we set d(a, b) = 0. For technical reasons, we also set $d(a, a^+) = d(a, a) = 0$ for each $a \in H$. The values d(a, b) for pairs $(a, b) \in T$ with $|H(a, b)| \geq 3$ are computed in the order of increasing value of |H(a, b)|. Thus, we first compute d(a, b) for all pairs $(a, b) \in T$ with |H(a, b)| = 3, then for all pairs $(a, b) \in T$ with |H(a, b)| = 4, etc.. For each pair d(a, b), we proceed using the following recursion rule.

For a given $(a, b) \in T$ with $|H(a, b)| \geq 3$, let $d = \max\{d(a, c) + d(c^+, b) \mid c \in H(a, b) \cup \{a\}\}$.

Note that no diagonal in R (a, b) is counted twice by taking d(a, c) +d(c^+, b) in the recursion, as R(a, c) and R(c^+, b) are disjoint. We show how to obtain d(a, b) from d. According to Observation 4 (i), the number d(a, b) equals d. In this remaining case, however, d(a, b) equals d + 1 by Observation 4(ii), where the +1 comes from the additional diagonal ab. Note in this case that, by Definition 3, Δ(f) = 0 also for the induced face f in R(a, b) that contains ab; hence Δ(f^0) = 0 for all induced faces f^0. For this reason and since every diagonal different from ab contains exactly two vertices of H(a, b), we deduce 2d = −Δ(a, b).

This gives the following case distinction for computing d(a, b): If d = d(a^+, b^-) and Δ(a, b)+ 2d = 0, we can add the diago nal ab to the d(a^+, b^-) diagonals and, thus, set d(a, b) = d + 1. Otherwise, we set d (a, b) = d.

The computation of d (a, b) takes time O (n) for each pair (a, b) ∈ T. Clearly, all the computation so far can be done in time O (n^3) by dynamic programming. After having computed d (a, b) for all pairs (a, b) ∈ T in this way, we check if there is a pair (a, b) ∈ T satisfying 2(d (a, b) + d (b, a)) ≥ h − 3i.

Lemma 10. If h > 3 then P admits a 2-connected cubic plane graph if and only if 2(d(a, b) + d(b, a)) ≥ h − 3i for some pair (a, b) ∈ T .

Proof: If P admits a 2-connected cubic plane graph, then, there is a diagonal configuration C on P with diagonal set D = D(C) such that Δ(f) = 0 for every induced face f ∈ F (D). Diagonal ab of D is selected. As per the definition of d (a, b), the number of diagonals of D covered in R(a, b) is at most d(a, b). By Lemma 3, 2(d (a, b) + d(b, a)) ≥ |2|D| ≥ h − 3i.

On the other hand, suppose now that 2(d(a, b) + d(b, a)) ≥ h − 3i for some pair (a, b) ∈ T. Consider the two sets, D_1 and D_2, having the values of d(a, b) and d(b, a), respectively. By definition of d(a, b), the set $D_1 \cup D_2$ does not induce any unmatched vertex. If the diagonal ab lies in both D_1 and D_2, each face induced by $D_1 \cup D_2$ satisfies Δ(f) = 0 and we can find a 2-connected cubic plane graph on P due to Lemma 8.

Suppose now that the diagonal ab does not lie in $D_1 \cap D_2$. If it lies neither in D_1 nor in D_2 then Δ(f) = 0 holds for each face induced by the set $D_1 \cup D_2$, with the possible exception of the face. Δ(f) = 0 holds for each face induced by the set $D_1 \cup D_2$, with the possible exception of one face adjacent to the diagonal ab. Due to Lemma 3, removing consequently h-3i-(d(a, b) + d(b, a)) The maximizing function of the dynamic program is then extended with the two parameters s, t ∈ {0, 1, 2, 3} and the additional parameter j ∈ {0, 1}, which is 1 if the diagonal ab is contained in C else it is 0. The following is inferred for the function:

(i) D contains the diagonal ab if and only if j = 1.
(ii) D − ab is the diagonal set of a diagonal configuration C on P ∩ R(a, b), with the only exception that the degrees of a and b in this configuration are 3 − s and 3 − t (instead of 3, as in Definition 1).
(iii) Every face f^0 that is induced by D satisfies Δ(f^0) = 0, with the possible exception of the unique face f intersecting the complement of R(a, b).

If such set D does not exist, z(a, b, s, t, j) is undefined. z(a, a^+, 2, 2, 1) = 1. If aa^+ is not present in D, z(a, a^+, 3, 3, 0) := 0. Other values z(a, a^+, s, t, j) are undefined. In computing the values of z(a, b, s, t, j), use of the following analogue of Observation 2 is used:

Observation 2. Let (a, b) ∈ T and let D be the set of diagonals as defined in. Let f be the unique face induced by D that contains the segment ab and some point to the right of the segment ab. Then f contains a point c ∈ H(a, b). Consequently, every diagonal of D −ab lies entirely in exactly one of the closed half spaces R (a, c) and R(c, b).

We now consider the general case of computing a value z(a, b, s, t, j) for |H(a, b)| > 0 and this value may turn out to be indeterminate. Due to Observation 2.

z(a, b, s, t, j) = max{j + z(a, c, s^0, t^0, j^0) + z(c, b, s^{00}, t^{00}, j^{00})},

Where the maximum is taken over all c ∈ H(a, b), all s^0, s^{00}, t^0, t^{00} ∈ {0, 1, 2, 3} and all j^0, j^{00} ∈ {0, 1}, for which the following five conditions are satisfied:

1. z(a, c, s^0, t^0, j^0) and z(c, b, s^{00}, t^{00}, j^{00}) are defined (consistency)
2. s + j ≤ s^0 (degree condition at a)
3. t + j ≤ t^{00} (degree condition at b)
4. s^{00} + t^0 ≥ 3 (degree condition at c)
5. if j = 1, then 3i = h − 2d + (2 − s) + (2 − t).

Condition (1) ensures that only defined values are propagated in the dynamic programming approach. Condition (2) (and, analogously, Condition (3)) ensure that the s + j (t + j) half-edges on a (on b) that either leave R(a, b) or belong to the diagonal ab are a subset of the s^0 (t^{00}) half-edges on a (on b) leaving R(a, c) (R(c, b)). Note that the inequality cannot be replaced by equality, since we have to allow possible half-edges on a that are fully contained in R(a, b) but not fully contained in R(a, c) (analogously for half-edges on b).

Similarly, there may be half-edges on c that are neither fully contained in R(a, c) nor R(c, b); each of these contributes +1 to both numbers s^{00} and t^{0}. Thus, we have to allow that $s^{00} + t^{0} > 3$; however, since c has exactly three incident half-edges and since $s^{00} + t^{0}$ covers each of these edges, $s^{00} + t^{0}$ cannot be lower than 3, which gives Condition (3).

In exactly the same way as in the previous algorithm, a pair (a, b) ∈ T is chosen that maximizes the total number of diagonals to the left and right of the segment ab. Using the pre computed values of z reconstruct a graph G. G satisfies Δ (f^{0}) = 0 for all induced faces f^{0} except for at most one face f that is incident to a or b and has Δ (f) > 0. Unmatched vertices were not forbidden so far, and G may contain some unmatched vertices. After applying Operations 1and 2, a diagonal configuration is obtained having no unmatched vertex which satisfies Δ(f) = 0 for each induced face. The construction of the desired plane cubic graph is completed by applying Lemma 7.

6. References

1. Garcia A, Hurtado F, Huemer C, Tejel J and Valtr P. On tri connected and cubic plane graphs on given point sets, Compute. Geom. Theory Appl. 2009; 42:913–22.
2. Garcia A, Hurtado F, Huemer C, Tejel J and Valtr P. On embedding tri connected cubic graphs on point sets. Electronic Notes in Discrete Mathematics. 2007; 29: 531–8.
3. Olaverr AG, Hurtado IF, Huemer C, Tejel J and Valtr P. On tri connected and cubic plane graphs on given point sets. Comput Geom. 2009; 42(9):913–22.
4. Tamura A, Tamura Y. Degree constrained tree embedding into points in the plane. Inf Process Letter. 1992; 44:211–4.
5. Jackson B, Jordan T. Independence free graphs and vertex connectivity augmentation. J Combine Theory B. 2005; 94:31–77.
6. Katz B, Krug M, Rutter I, Wolff A. Manhattan-Geodesic embedding of planar graphs. Proceedings of Graph Drawing; 2009. p. 207–18.
7. Lo CY, Matousek J, Steiger W. Algorithms for ham-sandwich cuts. Discrete and Computational Geometry. 1994; 11:433–52.
8. Mondal D, Nishat RI, Rahman MS, Alam MJ. Minimum-area drawings of plane 3-trees, Journal of Graph Algorithms and Applications. 2011; 15(2):177–204.
9. Kant G. Augmenting outer planar graphs. J Algorithms. 1996; 21:1–25.
10. Bernardo M, Abrego, Fernandez-Merchant S. A lower bound for the rectilinear crossing Number Graphs and Combinatorics. 2005; 21(3):293–300.
11. Kurowski M. A 1.235 lower bound on the number of points needed to draw all n –vertex planar graphs. Inf Process Letters. 2004; 92(2):95–8.
12. Gemignani M. On finite subsets of the plane and simple closed polygonal paths, Mathematics Magazine. 1966; 39(1):8–48.
13. Geetha NK. Combinatorial theory of a complete graph K5. IOSR Journal of Mathematics. 2014; 5:10–12.
14. Geetha NK. The intrinsic parity phase factors of regular graphs. International Journal of Advanced Research. 2014; 2:1087–90.
15. Aichholzer O, Krasser H. The point set order type data base: A collection of applications and results. Proceedings of 13th Canada Conference on Computational Geometry; Waterloo, Canada. 2001. p. 17–20.
16. Bose P, McAllister M, Snoeyink J. Optimal algorithms to embed trees in a point set. J Graph Algorithms Appl. 1997; 1(2):1–15.
17. Bose P, Morin P. Competitive online routing in geometric graphs. Theoretical Computer Science. 2004; 324:273–88.
18. Bose P, Dujmovic V, Hurtado F, Langerman S, Morin P, Wood DR. A polynomial bound for untangling geometric planar graphs. Discrete and Comp Geom. 2009; 24(4): 570–85.
19. Brass Pe, Cenek E, Duncan Cristian A, Efrat A, Erten C, Ismailescu DP, Kobourov Stephen G, Lubiw Anna, Joseph Mitchell SB. On simultaneous planar graph embedding's. Compute Geom Theory Appl. 2007; 36(2):117–30.
20. Gritzmann P, Mohar B, Pach J, Pollack R. Embedding a planar triangulation with vertices at specified points. Amer Math Monthly. 1991; 98(2):165–6.
21. Nishat RI, Mondal D, Rahman MS. Point-set embedding's of plane 3-trees. Compute. Geom: Theory and Appl. 2012; 45(3):88–98.
22. Cabello S. Planar embed ability of the vertices of a graph using a fixed point set is NP-hard. Journal of Graph Algorithms and Applications. 2006; 10(2):353–63.
23. Dey TK, Dillencourt MB, Ghosh SK, Cahill JM. Triangulating with high connectivity. Compute Geom Theory and Appl. 1997; 8(1):39–56.
24. Moosa TM, Rahman MS. Improved algorithms for the point-set embed ability problem for plane 3-trees. 17th International Computing and Combinatorics Conference (COCOON 2011); 2011. p. 204–12.