Indian Journal of Science and Technology, *Vol 8(13), 53160, July 2015*
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Algorithms to Find Geodetic Numbers and Edge Geodetic Numbers in Graphs

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Abstract

For any two vertices u and v of a graph G = (V, E), any shortest path joining u and v is called a u-v geodesic. Closed interval I[u,v] of u and v is the set of those vertices belonging to at least one u-v geodesic. A subset S of V(G) is an geodetic set if every vertex of G lies in at least one closed interval between the vertices of S. The geodetic set of a minimum cardinality in G is called as minimum geodetic set. The cardinality of the minimum geodetic set is the geodetic number of G denoted by gn(G). For a non-trivial connected graph G, a set S ofV (G) is called an edge geodetic cover of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number egn (G) of G is the minimum order of its edge geodetic covers and any edge geodetic cover of order egn(G) is an edge geodetic basis. This paper introduces the algorithms to find geodetic numbers and edge geodetic numbers in connected graphs using dynamic programming approach.

Keywords: Diameter, Distance, Eccentricity, Edge Geodetic Cover, Edge Geodetic Number, Geodesic, Geodetic Number, Geodetic Set, Radius

1. Introduction

A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G, called edges. The vertex set and the edge set of G are denoted by V(G) or simply V and E(G) or E respectively. A graph G is said to be connected if any two distinct vertices of G are joined by a path. The distance d(u,v) between two vertices u and v in G is the length of a shortest u-v path in G. A u-v path of length d(u,v) is called a u-v geodetic. The eccentricity e(u) of a vertex u is defined by e(u) = max{d(u,v):vϵV}. The radius r and diameter d of G are defined by r = min {e(v) : vϵV} and d = max {e(v): v ϵV} respectively. For graph theoretic terminology, we refer to[1,2].

1.1 Geodetic Number

For two vertices u and v in a connected graph G, the closed interval I[u, v] of two vertices u and v in G is the set of those vertices of G belonging to at least one u-v geodesic that is the shortest path between u and v. Note that a vertex w belongs to I[u,v] if and only if there is a shortest path between (u,w) and (w,v). For a set S of vertices, let the closed interval I[S] of S be the union of the closed intervals I[u, v] over all pairs of vertices u and v in S. A set of vertices S is called geodetic if I[S] contains all vertices V of G. Harary et al[3] define the geodetic number gn(G) of a graph G as the minimum cardinality of a geodetic set. The calculation of the geodetic number is an NP-hard problem for general graphs[4,9]. The concept of geodetic number of a graph was introduced in[3,5,6] and its different dimensions and types are further studied in[7,8,10–16]. Let us take for example the graph given in Figure 1 to identify the geodetic set. Consider S= {u_1, w_3}. The distance between u_1 and w_3 can be measured as d(u_1,w_3) = 3.

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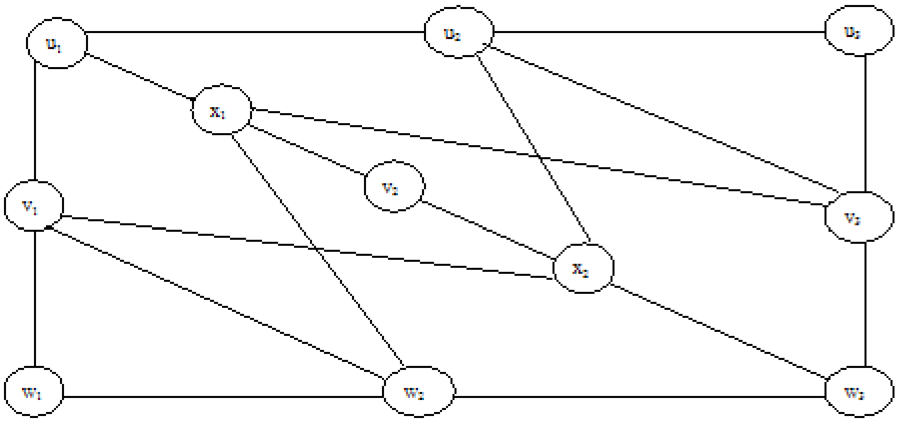

Figure 1.

The closed interval $I[u_1,w_3] = \{ u_1, u_2, v_1, v_3, w_2, w_3 ,x_1, x_2\}$, $I[S] = V(G) - \{ u_3,v_2,w_1\}$ and hence it does not form the geodetic set. When we take $S= \{u_3, v_2, w_1\}$, $I[S] = \{u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3 , x_1, x_2\}$. $I[S] =V(G)$ and hence $S= \{u_3, v_2, w_1\}$ is the minimum geodetic set. As per Harary et al.[3], the cardinality of S is geodetic number and hence geodetic number of the graph G is gn(G)=3.

1.2 Edge Geodetic Number

For two vertices u and v in a connected graph G, the closed interval $I_e[u, v]$ of two vertices u and v in G is the set of those edges of G belonging to at least one u-v geodesic that is the shortest path between u and v. Note that the edges (u,w) and (w,v) belongs to $I_e[u,v]$ if and only if there is a shortest path between (u,w) and (w,v). For a set S of vertices, let the closed interval $I_e[S]$ of S be the union of the closed intervals $I_e[u, v]$ over all pairs of vertices u and v in S. A set of vertices S is called edge geodetic if $I_e[S]$ contains all edges E of G, that is, $I_e[S] = E(G)$. Concepts of edge geodetic number of a graph and related results have been studied from [10–12]. Edge geodetic number egn(G) of a graph G is defined as the minimum cardinality of an edge geodetic set.

Consider the graph G given in Figure 2 and $S = \{u_1, u_3, u_5\}$. The closed interval is evaluated as $I_e[S] = \{(u_1, u_2), (u_1, u_4), (u_2, u_3), (u_3, u_4), (u_2, u_5), (u_3, u_5), (u_4, u_5)\}$. It is found that Ie[S] =E(G) and hence S is an edge geodetic basis of G so that egn (G) = 3. Consider a set $S= \{u_2, u_3, u_4, u_5\}$ is an edge geodetic set of G and it is clear that no proper subset of S is an edge geodetic set of G and so S is a minimal edge geodetic set of G. Every minimum edge geodetic set of G is a minimal edge geodetic set of G and the converse is not true.

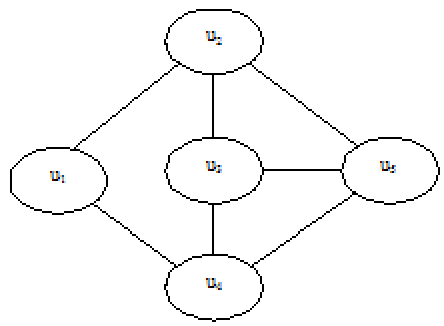

Figure 2.

For the graph G given above, $S= \{u_2, u_3, u_4, u_5\}$ is a minimal edge geodetic set but not a minimum edge geodetic set of G. A geodetic graph of order p with edge geodetic number p is characterized. It is shown in [10] that every pair k, p of integers with 2≤k≤p is realizable as the edge geodetic number and order of some connected graph. For positive integers r, d and k≥2 with r<d≤2r, there exists a connected graph of radius r, diameter d and edge geodetic number k. It is shown that if G is a geodetic graph of order p and diameter d, then egn(G)≤p−d+1. Also, for integers p, d and k with 2≤d<p, 2≤k<p and p−d−k+1≥0, there exists a graph G of order p, diameter d and edge geodetic number k. The geodetic number gn(G) of G is related to egn(G) as gn(G)≤ egn(G). Further, it is shown that for any positive integers 2≤a≤b, there exists a connected graph G such that gn(G)=a and egn(G)=b.

2. Algorithm for Computing Geodetic Number

In this section we introduce an algorithm using dynamic programming approach for evaluating the geodetic number of a connected graph G. The overall process is divided into two divisions in the algorithm. The first division of the algorithm records the shortest path between each vertex to every other vertices for further process using all source shortest path algorithm introduced by Floyds. The second part of the algorithm uses the recorded shortest path and finds the closed interval I[S] which is the union of the intervals I [u, v] over all pairs of vertices u and v and thereby determine the geodetic number of a given graph G.

2.1 Algorithm for Geodetic Number

Algorithm GeodeticNumber(G)
// Input: An undirected graph G=(V,E) |V| ≥2
// Output: Geodetic Number of G

1. Using Floyd's All source shortest path algorithm, find the shortest path between each vertex and every other vertices and vertices in the shortest path are recorded in a set named Path[u,v].
2. SSize = 1
3. Found = false
4. While (SSize ≤ |V| and Found≠ true)
5. SSize = SSize+1
6. Generate SS subsets of V such that V S϶ SS and |S|=SSize
7. While SS≠Ø and Found≠ true
8. Get next subset S from SS
9. For every vertex u and v in S
10. Find the closed interval I[S] union of the intervals I[u, v] over all pairs of vertices u and v in S from set Path[u, v] computed above in Step 1.
11. Loop
12. If I[S]=V Then Found=true
13. Loop
14. Loop
15. Return SSize

Theorem 2.1: For any undirected connected graph G such that|V|≥2, the algorithm 2.1 finds the geodetic number gn(G) and 2≤ gn(G) ≤|V|.n

Proof: Let G be a connected graph with V={v_1,v_2,v_3,……,v_n}. Step 1 of the Algorithm 2.1 evaluates the shortest path between each vertex and every other vertex. Step 6 of the Algorithm 2.1 generates subsets of V, SS = {v_1, v_2, v_3,……,v_k | 2≤k≤n}. Cardinality of generated subset starts with minimum value 2 and cardinality of the subset increases by one during each iteration until geodetic set is found. Step 8 of the Algorithm 2.1 chooses a subset S = {v_1, v_2, v_3,……,v_k | 2≤k≤n } from SS and Step 10 of the Algorithm 2.1 executes the main the process of finding the geodetic number by evaluating closed interval I[S] which has the set of vertices of G that forms shortest path between vertices in S from the Path set that contains the shortest path between each vertex and every other vertex. As per Step 12 of the Algorithm 2.1, if I[S]=V(G) then S is considered as the minimal geodetic set and returns [S]≥2 to be the geodetic number. Steps 4 to step 14 of the Algorithm 2.1 are repeated until a minimum geodetic set is identified.

If I[S] = V(G) in the first iteration itself, the geodetic number is 2 otherwise it linearly increases by one during each iteration. For the worst case S contains all the vertices of G and so the geodetic number gn(G) is |V| Hence it is proved that 2≤ gn(G) ≤|V|.

Theorem 2.2: Geodetic number of an undirected graph G can be computed using Algorithm 2.1 in O(n^3 +2^n).

Proof: Let G be a connected undirected graph with n vertices, V= {v_1, v_2, v_3, … v_n}. Algorithm 2.1 initiates the task of finding geodetic number by evaluating all source shortest path and records in a set named Path using the Floyd's algorithm which is proved having the efficiency O(n^3) and records the path in a Set data structure and referred later. The process continues by constructing the subset of V(G) that is S (as per Step 6 of Algorithm 2.1) and evaluating the closed interval I[S] of S be the union of the intervals I[u, v] over all pairs of u-v geodesic of vertices u and v in S. The computing of I[S] which is the geodetic closure of a set of vertices is done by referring the Path set constructed in Step 1 of the Algorithm 2.1. For each subset S, computing of I[S] is to be performed repeatedly until the geodetic set is finalized. There are maximum 2^n subsets constructed from the vertices set and the closed interval evaluation is performed for the worst case for all the subsets there by this process efficiency is O(2^n). Hence the computing of geodetic number gn(G) using Algorithm 2.1 can be completed in O(n^3+2^n).

3. Algorithm for Computing Edge Geodetic Number

We introduce an algorithm using dynamic programming approach in this section for evaluating the edge geodetic number of a connected graph G. The overall process is divided into two divisions as in the Algorithm 2.1. The first division of the algorithm records the edges that fall in the shortest path between each vertex to every other vertex for further process using Floyd's all source shortest path algorithm. The second part of the algorithm uses the recorded shortest path and finds the closed interval $I_e[S]$ which is the union of the intervals $I_e[u, v]$ over all pairs of vertices u and v and thereby determine the edge geodetic number of the given graph G.

3.1 Algorithm for Edge Geodetic Number

Algorithm EdgeGeodeticNumber(G)
// Input: An undirected graph G=(V,E) |V| ≥2
// Output: Edge Geodetic Number of G
1. Using Floyd's All source shortest path algorithm, find the shortest path between each vertex and every other vertices and edges in the shortest path are recorded in a set named Path[u,v].
2. SSize = 1
3. Found = false
4. While (SSize ≤ |V| and Found≠ true)
5. SSize = SSize+1
6. Generate SS subsets of V such that V Sє SS and |S|=SSize
7. While SS≠Ø and Found≠ true
8. Get next subset S from SS
9. For every vertex u and v in S
10. Find the closed interval $I_e[S]$ union of the intervals I_e [u, v] over all pairs of vertices u and v in S from set Path[u, v] computed above in Step 1.
11. Loop
12. If $I_e[S]$ = E Then Found=true
13. Loop
14. Loop
15. Return SSize

Theorem 3.1: For any undirected connected graph G such that |V|≥2, the algorithm 3.1 finds the edge geodetic number egn(G) and 2≤ egn(G) ≤|V|

Proof: Let G be a connected graph with $V=\{v_1,v_2,v_3,\ldots\ldots,v_n\}$. Step 1 of the Algorithm 3.1 evaluates the shortest path between each vertex and every other vertex. Step 6 of the Algorithm 3.1 generates subsets of V, $SS = \{v_1, v_2, v_3,\ldots\ldots,v_k \mid 2\le k\le n\}$. Cardinality of generated subset starts with minimum value 2 and cardinality of the subset increases by one for each iteration until edge geodetic set is found. Step 8 of the Algorithm 3.1 chooses a subset $S = \{v_1, v_2, v_3,\ldots\ldots,v_k \mid 2\le k\le n \}$ from SS and Step 10 of the Algorithm 3.1 executes the main the process of finding the edge geodetic number by evaluating closed interval $I_e[S]$ which has the set of edges of G that forms shortest path between vertices in S from the Path set that contains the edges fall in the shortest path between each vertex and every other vertex in S. As per Step 12 of the Algorithm 3.1, if $I_e[S]$=E then S is considered as the minimal edge geodetic set and returns |S|≥2 to be the edge geodetic number. Steps 4 to step 14 of the Algorithm 3.1 are repeated until a minimum edge geodetic set is identified.

If $I_e[S]$=E in the first iteration itself then the edge geodetic number is 2 otherwise it linearly increases by one during each iteration. For the worst case S contains all the vertices of G and so the edge geodetic number egn(G) is |V|Hence it is proved that 2≤ egn(G) ≤|V|.

Theorem 3.2: Edge Geodetic number of an undirected graph G can be computed using Algorithm 3.1 in $O(n^3+2^n)$.

Proof: Let G be a connected undirected graph with n vertices, $V= \{v_1, v_2, v_3, \ldots v_n\}$and the edges $E= \{e_1, e_2, e_3, \ldots$ en}. Algorithm 3.1 initiates the task of finding edge geodetic number by evaluating all source shortest path using the Floyd's algorithm which is proved having the efficiency $O(n^3)$ and records the path in a Set data structure where each element is the edge that falls in the shortest path and this set is referred later. The process continues by constructing the subset of V(G) that is S (as per Step 6 of Algorithm 3.1) and evaluating the closed interval $I_e[S]$ of S be the union of the intervals $I_e[u, v]$ over all pairs of u-v geodesic of vertices u and v in S . Computing of $I_e[S]$, the edge geodetic closure of a set of edges is done by referring the Path set constructed in Step 1 of the Algorithm 3.1. For each subset S, computing of $I_e[S]$ is to be performed repeatedly until finalizing the edge geodetic set. There are maximum 2^n subsets possibly constructed from V and so the closed interval evaluation is performed for the worst case for all the subsets there by this process efficiency is $O(2^n)$. Hence the computing of edge geodetic number egn(G) using Algorithm 3.1 can be completed in $O(n^3+2^n)$.

4. Applications

There are interesting applications of Geodetic number concepts to the problem of designing the route for a shuttle and communication network design.

The edge geodetic set is more advantageous to the real life application of routing problem. In particular, the edge geodetic sets are more useful than geodetic sets in the case of regulating and routing the goods vehicles to transport the commodities to important places. The different other areas that apply geodetic number concepts are telephone switching centres, facility location, distributed computing, information retrieval, Neural networks and Data mining.

5. Conclusion

There was an algorithm proposed by Nancy Kinnersely and M.C.Kong for computing geodetic basis whose efficiency is $O(n^3X2^n)$. In this paper we have demonstrated the algorithms for computing the geodetic number and edge geodetic number of undirected graphs using dynamic programming approach and analyzed its efficiency which is $O(n^3+2^n)$ Delivering of an algorithm with improved efficiency may be highlighted to be the novelty of this paper and this algorithm may be applied for solving problems related to the applications mentioned in the previous section. There are different types of Geodetic Numbers possibly studied for which algorithms can be developed.

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