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Average D-distance Between Edges of a Graph

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Abstract

The D-distance between vertices of a graph G is obtained by considering the path lengths and as well as the degrees of vertices present on the path. The average D-distance of a connected graph is the average of the D-distance between all pairs of vertices of the graph. Similarly, the average edge D-distance is the average of D-distances between all pairs of edges in the graph. In this article we study the average edge D-distance of a graph. We find bounds for average edge D-distance which are sharp and also prove some other results.

Key words: Average D-distance, D-distance, Diameter, 2000 Mathematics subject classifications: 05C12

1. Introduction

The concept of distance is one of the important concepts in study of graphs. It is used in isomorphism testing, graph operations, hamiltonicity problems, extremal problems on connectivity ad diameter, convexity in graphs etc. Distance is the basis of many concepts of symmetry in graphs.

In addition to the *usual distance*, $d(u,v)$, between to vertices $u,v\in V(G)$, we have *detour distance*[1], *superior distance*[5], *signal distance*[7], degree distance etc.

In an earlier article[9], the authors introduced the concept of *D-distance* by considering not only path length between vertices, but also the degrees of all vertices present in a path while defining the D-distance. In a natural way we can extend this concept to D-distance between edges also.

Also we have the concept of average distance in graphs which was introduced by Dankelmann[3–5]. In[10], we studied the average distance between vertices of a graph with respect to D-distance. In this article, we study the average D-distance between edges.

The article is arranged as follows. In §2, we collect some definitions and results for easy reference. In §3, we study some properties of average edge D-distance and in §4, we calculated the average D-distance between edges for some classes of graphs.

2. Preliminaries

Throughout this article, by a graph $G=G(V, E)$, we mean a non-trivial, finite, undirected graph without multiple edges and loops. Unless otherwise specified, all graphs we consider are connected. For any unexplained notation and terminology, we refer [1].

In this section we given some definitions and state some results for later use. We begin with D-distance in graphs.

2.1 Definition 1

In a graph G, the *degree of a vertex* v, $\deg(v)$, is the number of edges which are incident with v .

Similarly we can define the *degree of an edge* $e=(u,v)$ as the number of edges which have a common vertex with the edge e i.e $\deg(e) = \deg(v) + \deg(u) - 2$.

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2.2 Definition 2

For any connected graph G, we define

$\Delta(G) = \max\{\deg(v) : v \in V(G)\}$ as the *maximum vertex degree* of G.

$\delta(G) = \min\{\deg(v) : v \in V(G)\}$ as the *minimum vertex degree* of G.

$\Delta^1(G) = \max\{\deg(e) : e \in E(G)\}$ as the *maximum edge degree* of G.

$\delta^1(G) = \min\{\deg(e) : e \in E(G)\}$ as the *minimum edge degree* of G.

2.3 Definition 3

If u,v are vertices of a connected graph G, the *D-length* of a connected $u - v$ path s is defined as $l^D(s) = l(s) + \deg(v) + \deg(u) + \sum \deg(w)$ where sum runs over all intermediate vertices w of s and $l(s)$ is the length of the path.

2.4 Definition 4

The *D-distance*, $dD\ (u,v)$ between two vertices u,v of a connected graph G is defined as $dD(u,v) = \min\{lD(s)\}$ where the minimum is taken over all $u - v$ paths s inv G In other words, $d^D(u,v) = \min\left\{l(s) + \deg(u) + \deg(v) + \sum \deg(w)\right\}$ where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths s in G.

If u,v are two vertices of a graph, then $d(u,v)$ denotes the usual distance between u and v. By $e = e(u,v)$ in $E(G)$, we mean an edge adjacent with the vertices u and v.

2.5 Definition 5

Let G be a connected graph and let $e(u_1,v_1)$ and $f(u_2,v_2)$ be two edges of G. The *D-distance* between these edges is defined as $ed^D(e,f) = \min\left\{d^D(u_1,v_1), d^D(u_1,v_2), d^D(u_2,v_1), d^D(u_2,v_2)\right\}$.

2.6 Remark

Observe that $ed^D(e_1,e_2) = 0 \Leftrightarrow e_1,e_2$ are *neighbor edges* i.e., they have one common vertex.

2.7 Definition 6

Let G be a connected graph of order n. The *average distance* of G, denoted by $\mu(G)$, is defined as $\mu(G) = \binom{n}{2}^{-1} \sum_{\{u,v\}\subset V} d(u,v)$ where $d(u,v)$ denotes the distance between the vertices u and v[3–5].

Similarly, we can define the average D-distance of a graph as follows:

2.8 Definition 7

Let G be a connected graph of order n. The *average D-distance between vertices of G*, denoted by $\mu D(G)$, is defined as $\mu^D(G) = \binom{n}{2}^{-1} \sum_{\{v,u\}\subset V} d^D(u,v)$ where $dD(u,v)$ denotes the *D-distance* between the vertices u and v.

Similarly, we can define[8] the average D-distance between edges of a graph as follows:

2.9 Definition 8

Let G be a connected graph of order n. The *average D-distance between edges of G*, denoted by $\mu_3^D(G)$, is defined as $\mu_3^D(G) = \binom{n}{2}^{-1} \sum_{\{e_1,e_2\}\subset E} ed^D(e_1,e_2)$ where $ed^D(e_1,e_2)$ denotes the *D-distance* between the edges e_1 and e_2.

Some more definitions.

2.10 Definition 9

A *spanning subgraph* is a subgraph of G that contains all the vertices of G.

2.11 Definition 10

Let G be a connected graph of order n having m edges with $V(G) = \{v_1,v_2,..,v_n\}$. The *D-distance matrix* of G, denoted as $D^D(G)$, is defined as $D^D(G) = [d_{i,j}^D]_{n\times n}$ where $d_{i,j}^D = d^D(v_i,v_j)$ is the D-distance between the vertices *vi* and *vj*.

Obviously $D^D(G)$ in a $n \times n$ symmetric matrix with all diagonal entries being zero.

In a similar manner we can define edge *D-distance matrix* [*EDDM*] of G, denoted as $D_3^D(G)$, is defined as $D_3^D(G) = [d_{i,j}^D]_{m\times m}$ where $ed_{i,j}^D = ed^D(e_i,e_j)$ is the edge D-distance between the edges e_i and e_j and $d_{i,i}^D = \infty$. Thus this is a $m \times m$ symmetric matrix.

Further, we have

2.12 Definition 11

Let G be a graph, then the *average degree* of G, denoted as $d(G)$, is given by $d(G)=\frac{1}{|V|}\sum_{v\in V} d(v)$ where $d(v)$ is the degree of the vertex v.

2.13 Definition 12

The *total edge D-distance* $[TEDD]$ of graph G is the number given by $\frac{1}{2}\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} ed^{D}(e_i,e_j)$ where m is the number of edges.

3. Average Edge D-distance

In this section we prove some results on average D-distance between edges.

3.1 Theorem 1

Let G_1 and G_2 be two connected graphs having same orders and same diameters. If the number of edges in G_1 is more than the number of edges in G_2 then average edge D-distance of G_1 is more than average edge D-distance of G_2.

Proof: Since the diameters of these two graphs are the same, the largest entries in the edge D-distance matrix of these graphs are the same. The number of the pairs of edges examined is greater in the graph whose edge number is greater. And this cause *TEDD* value to increase. Since these orders are same and the number of edges in G_1 is more than number of edges in G_2 then average edge D-distance of G_1 is more than average edge D-distance of G_2 i.e $|E(G_1)| > |E(G_2)| \Rightarrow \mu_3^D(G_1) > \mu_3^D(G_2)$.

3.2 Theorem 2

Let G_1 and G_2 be two connected graphs of same order and $diam\,D\,(G1) < diam\,D\,(G2)$, Then $\mu_3^D(G_1) > \mu_3^D(G_2)$.

Proof: Since G_1,G_2 have same number of vertices and *$diam(G_1) < diam(G_2)$, it is clear that* $|E(G_1)| > |E(G_2)|$. Then by theorem 3.1, we have $\mu_3^D(G_1) > \mu_3^D(G_2)$.

3.3 Theorem 3

Let G_1 and G_2 be two connected graphs having same orders and same diameters. If $\delta^1(G_1) < \delta^1(G_2)$ then $\mu_3^D(G_1) < \mu_3^D(G_2)$.

Proof: Since $\delta^1(G_1) < \delta^1(G_2)$ and $|V(G_1)| = |V(G_2)|$, we have $|E(G_1)| < |E(G_2)|$. Then by theorem 3.1 $\mu_3^D(G_1) < \mu_3^D(G_2)$.

3.4 Theorem 4

Let G_1 and G_2 be two connected graphs having same orders and diameters. If $\delta(G_1) < \delta(G_2)$ then $\mu_3^D(G_1) < \mu_3^D(G_2)$.

Proof: $\delta(G_1) < \delta(G_2)$ implies $|E(G_1)| < |E(G_2)|$ then by theorem 3.1 $\mu_3^D(G_1) < \mu_3^D(G_2)$.

3.5 Theorem 5

Let G_1 and G_2 be two connected graphs having same orders and same diameters. If the average degree of G_1 is less than average degree of G_2 then $\mu_3^D(G_1) < \mu_3^D(G_2)$.

Proof: We have by definition $|E| = \frac{1}{2}\sum_{v\in V} d(v) = \frac{1}{2} d(G)|V|$. As the graphs have same order, if $d(G_1) < d(G_2)$, then $|E(G_1)| < |E(G_2)|$. Hence by theorem 3.1, we have $\mu_3^D(G_1) < \mu_3^D(G_2)$.

3.6 Theorem 6

Let H be a spanning subgraph of G. Then $\mu_3^D(H) < \mu_3^D(G)$.

Proof: Number of the vertices of H will remain the same as the graph itself it is obvious that $|E(H)| < |E(G)|$. From theorem 3.1, we have $\mu_3^D(H) < \mu_3^D(G)$.

4. Results on Some Classes of Graphs

Here we calculate the average edge D-distance for some classes of graphs.

4.1 Theorem 1

If Pn is the path graph with $n(\geq 3)$ vertices and $n-1$ edges, then $\mu_3^D(P_n) = \frac{2a_n(n+1)}{n(n-1)}$ where an is given by the relation $a_n = a_{n-1} + n - 3$ with $a_3 = 0$.

Proof: For Pn, the edge D-distance matrix, $\mu_3^D(G)$, is the $n-1 \times n-1$ symmetric matrix

$$\begin{bmatrix} \infty & 0 & 5 & 8 & 11 & \cdots & 3n-13 & 3n-10 & 3n-7 \\ & \infty & 0 & 5 & 8 & \cdots & 3n-16 & 3n-13 & 3n-10 \\ & & \infty & 0 & 5 & \cdots & 3n-19 & 3n-16 & 3n-13 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & \infty & 0 & 5 & 8 & 11 \\ & & & & & \infty & 0 & 5 & 8 \\ & & & & & & \infty & 0 & 5 \\ & & & & & & & \infty & 0 \\ & & & & & & & & \infty \end{bmatrix}$$

By adding all entries in the upper triangular or lower triangular matrix we get the total edge D-distance, which is $an(n + 1)$ in this case, where an $n(\geq 3)$ is a constant given by {0, 1, 3, 6, 10, 15, 21,.....} or recursively $a_n = a_{n-1} + n - 3$.

Then $\mu_3^D(G) = \dfrac{2a_n(n+1)}{n(n-1)}$.

4.2 Theorem 2

For a complete graph Kn with n vertices, the average edge D-distance $\mu_3^D(K_n)$ *is give by* $\mu_3^D(K_n) = \dfrac{(n-2)(n-3)(2n-1)}{4}$.

Proof: Every edge taken from *Kn* has $2(n - 2)$ edge neighbors and $\binom{n-2}{2}$ distinct edges (due to end points of this edge). The edge D-distance between any edge and its neighbor is zero and the edge D-distance between any edge and distinct edge in *Kn* = $(2n - 1)$. Total edge D-distance $TEDD = \dfrac{1}{2}\binom{n}{2}\binom{n-2}{2}(2n-1)$ and

$$\mu_3^D(K_n) = \frac{TEDD}{\binom{n}{2}} = \frac{(n-2)(n-3)(2n-1)}{4}.$$

4.3 Theorem 3

The average edge D-distance of complete bipartite graph is

$$\mu_3^D(K_{m,n}) = \frac{mn(n-1)(m-1)(m+n+1)}{(m+n)(m+n-1)}.$$

Proof: In a complete bipartite graph there are *mn* edges. Any edge taken from complete bipartite graph has $(m + n - 2)$ edge neighbors and $(m - 1)(n - 1)$ distinct edges. The edge D-distance between any edge its neighbor is zero and the edge D-distance between any edge and distinct edge in complete bipartite graph is $(m + n + 1)$. $TEDD = \dfrac{mn(m-1)(n-1)(m+n+1)}{2}$.

Therefore $\mu_3^D(K_{m,n}) = \dfrac{mn(n-1)(m-1)(m+n+1)}{(m+n)(m+n-1)}$.

4.4 Theorem 4

The edge average D-distance of Star graph is $\mu_3^D(st_{1,n}) = 0$

Proof: For star graph one vertex is adjacent to all others. So every edge has one common vertex and total edge D-distance is zero therefore $\mu_3^D(st_{1,n}) = 0$

Alternatively, we may take m = 1 in theorem 4.3.

4.5 Theorem 5

The edge average D-distance of cyclic graph is $\mu_3^D(C_{2n}) = \dfrac{a_n}{2n-1}$ *where* $a_n = a_{n-1} + 6n - 5$ *with* $a_{2=5}$ *and* $\mu_3^D(C_{2n-1}) = \dfrac{(n-2)(3n+1)}{2(n-1)}$ $(n \geq 2)$.

Proof: *Case (i)* Cyclic graphs of odd order, $C_{2n-1}(n \geq 2)$

As *C2n* – 1 is regular, the elements of any row in the EDDM, except the diagonal element, are $0, 0, 5, 5, 8, 8, 11, 11, \ldots, 5 + \dfrac{3}{2}(2n-4)$. then

$$TEDD = \frac{2n-1}{2}\left[2(5+8+11+\ldots) + 5 + \frac{3}{2}(2n-5)\right].$$

$\mu_3^D(C_{2n-1}) = \binom{2n-1}{2}^{-1} TEDD$ and this can seen to be

$$\mu_3^D(C_{2n-1}) = \frac{(n-2)(3n+1)}{2(n-1)}.$$

Case (ii): Cyclic graphs of even order $C_{2n}(n \geq 2)$ In this case the elements of any row in EDDM except the diagonal element, are $0, 0, 5, 5, 8, 8, 11, 11, \ldots, 5 + \dfrac{3}{2}(2n-2), 5 + \dfrac{3}{2}(2n-2)$.

Like above we can show that $\mu_3^D(C_{2n}) = \dfrac{a_n}{2n-1}$ *where* $a_n = a_{n-1} + 6n - 5$ *with* $a_2 = 5$.

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