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Modular Multiplicative Divisor Labeling of Some Path Related Graphs

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Abstract

A graph G (V,E) with |V| = n is said to have modular multiplicative divisor labeling if there exist a bijection f: V(G) → {1, 2, ...,} and the induced function f*: E(G) → {0, 1, 2, ..., n – 1} where f*(uv) = f(u)f(v) (mod n) such that n divides the sum of all edge labels of G. We prove that the path P_n , and the graph $P_{a, b}$ (a graph which connects two vertices by means of "b" internally disjoint paths of length "a" each), shadow graph of a path and the cartesian product $P_n \times P_1$, (n is not a multiple of 6) admits modular multiplicative divisor labeling. Also we discuss the upper bound for the number of edges in a modular multiplicative divisor graphs.

AMS Subject Classification: 05C78

Keywords: Graph Labeling, Path, Graph $P_{a, b}$, Shadow Graph $D_2(G)$, The Cartesian Product $P_n \times P_1$

1. Introduction

For all terminology and notation in graph theory we follow Harary[4]. In this paper we consider only finite, simple, connected and undirected graphs. By a labeling we mean a one-to-one mapping that carries a set of graph elements on to a set of numbers, called labels (usually the set of integers). Graph labelings were first introduced in the mid sixties. Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. The qualitative labelings of graph elements have inspired research in diverse fields of human enquiry such as conflict resolution in social psychology, electrical circuit theory and energy crisis. Quantitative labelings of graphs have applications in coding theory problems including the design of good radar location nodes, missile guidance codes etc. An enormous body of literature has grown around the subject, especially in the last thirty five years and is still getting embellished due to increasing number of application driven concepts[5]. Also graph labeling is a strong communication between number theory[2] and structure of graphs[1]. Modular multiplication plays an important role in number theory problems[2]. Through some mathematical logic we could able to introduce a new labeling called modular multiplicative divisor (MMD) labeling and we proved[3] complete graph K_n, for all prime number n > 3, complete bipartite graph $K_{m,n}$, cycle graph C_n, n ≡ 1, 2(mod3) are modular multiplicative divisor graphs. Also we proved[9] split graph of cycle C_n, helm graph H_n, flower graph $f_{n\times 4}$, cycle cactus $C_{4(n)}$, extended triplicate graph of a path are modular multiplicative divisor graphs. Modular multiplicative divisor labeling techniques can be applied in the field of cryptography[7]. A pattern of ordering edge labels can be taken as the secret key in hashing the plain text for Authentication of the message. The present work is intended to discuss the existence of MMD labeling of a path and path related graphs and the maximum number of edges in a modular multiplicative divisor graphs.

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2. Definitions Related to the Context

Definition 2.1

A **graph labeling** is an assignment of integers to the vertices or edges, or both, subject to certain conditions.

Definition 2.2

A **path** in a graph is a sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. Edges and vertices appear only once in a path.

Definition 2.3

The **shadow graph** $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G" join each vertex v' in G' to the neighbours of the corresponding vertex v" in G".

Definition 2.4

Let u and v be two vertices. Connect u and v by means of "b" internally disjoint paths of length "a" each. The resulting graph is denoted by $\mathbf{P}_{a,b}$.

Definition 2.5

The cartesian product of two path graphs, one of which has only one edge is $P_n \times P_1$. It is called ladder graph L_n. It is a planar undirected graph with 2n vertices and n + 2(n – 1) edges.

3. Discussion of MMD Labeling of Path Related Graphs

In this section we consider the problem of whether the graphs in certain well-known and much-studied related graphs are modular multiplicative divisor. Here the discussion of MMD labeling of graphs under consideration is path and path related graphs (obtained through some graph operations).

Theorem: 3.1

Path P_n, n > 1 admits modular multiplicative divisor labeling.

Proof:

Let P_n be the path of n vertices $\{v_1, v_2, v_3, \ldots, v_n\}$ and n – 1 edges $\{e_1, e_2, e_3, \ldots e_{n-1}\}$.

Define a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ as follows.

Case (i) Let n = 4t + 2, t = 0, 1, 2, 3, … .

$$f(v_{n-4}) = \frac{3n-2}{4}, n > 2$$

$$f(v_{4i+1}) = i + 1, 0 \le i \le \frac{n-2}{4}$$

$$f(v_{4i+3}) = n - i - 1, 0 \le i \le \frac{n-6}{4}, n > 2$$

Sub case (i) when t = 0, 2, 4, ….

$$f(v_n) = n$$

$$f(v_{8i-4}) = \frac{n+2}{4} + i, 1 \le i \le \frac{n-2}{8}, n > 2$$

$$f(v_{8i}) = n - \frac{n+2}{4} - i, 1 \le i \le \frac{n-2}{8}, n > 2$$

$$f(v_{4i-2}) = \frac{3n+2}{8} + i, 1 \le i \le \frac{n-6}{4}, n > 2$$

Sub case (ii) when t= 1,3,5,….

$$f(v_n) = \frac{n}{2}; f(v_{n-2}) = n;$$

$$f(v_{8i-4}) = \frac{n+2}{4} + i, 1 \le i \le \frac{n-6}{8}, n > 6$$

$$f(v_{8i}) = n - \frac{n+2}{4} - i, 1 \le i \le \frac{n-6}{8}, n > 6$$

$$f(v_{4i-2}) = \begin{cases} \frac{3n-2}{8} + i, 1 \le i \le \frac{n-6}{8}, n > 6 \\ \frac{3n-2}{8} + 1 + i, \frac{n+2}{8} \le i \le \frac{n-6}{4}, n > 6 \end{cases}$$

Case (ii): Let n = 4t + 3, t = 0, 1, 2, 3, …

$$f(v_{4i+1}) = i + 1, 0 \le i \le \frac{n-3}{4}$$

$$f(v_{4i+3}) = n - i - 1, 0 \le i \le \frac{n-3}{4}$$

Sub case (i) when t = 0, 2, 4, ….

$$f(v_{n-1}) = n; f(v_{2+4i}) = \frac{3n+7}{8} + i, 0 \le i \le \frac{n-7}{4}, n > 3$$

$$f(v_{8i-4}) = \frac{n+1}{4} + i, 1 \le i \le \frac{n-3}{8}, n > 3$$

$$f(v_{8i}) = n - \frac{n+1}{4} - i, 1 \le i \le \frac{n-3}{8}, n > 3$$

Sub case (ii) when t = 1, 3, 5, ….

$$f(v_{n-3}) = n;\ f(v_{2+4i}) = \frac{3(n+1)}{8} + i,\ 0 \le i \le \frac{n-3}{4}$$

$$f(v_{8i-4}) = \frac{n+1}{4} + i,\ 1 \le i \le \frac{n-7}{8},\ n > 7$$

$$f(v_{8i}) = n - \frac{n+1}{4} - i,\ 1 \le i \le \frac{n-7}{8},\ n > 7$$

Case (iii): Let n = 4t + 4, t = 0, 1, 2, 3, …

$$f(v_{4i+1}) = i + 1,\ 0 \le i \le \frac{n-4}{4}$$

$$f(v_{4i+3}) = n - i - 1,\ 0 \le i \le \frac{n-4}{4}$$

Sub case (i) when t = 0, 2, 4, ….

$$f(v_n) = n\ ;\ f(v_{2+4i}) = \frac{3n+4}{8} + i,\ 0 \le i \le \frac{n-4}{4}$$

$$f(v_{8i-4}) = \frac{n}{4} + i,\ 1 \le i \le \frac{n-4}{8},\ n > 4$$

$$f(v_{8i}) = n - \frac{n}{4} - i,\ 1 \le i \le \frac{n-4}{8},\ n > 4$$

Sub case (ii) when t = 1, 3, 5, ….

$$f(v_n) = \frac{n}{2};\ f(v_{n-4}) = n$$

$$f(v_{8i-4}) = \frac{n}{4} + i,\ 1 \le i \le \frac{n-8}{8},\ n > 8$$

$$f(v_{8i}) = n - \frac{n}{4} - i,\ 1 \le i \le \frac{n-8}{8},\ n > 8$$

$$f(v_{2+4i}) = \begin{cases} \frac{3n}{8} + i, 0 \le i \le \frac{n-8}{8} \\ \frac{3n}{8} + 1 + i, \frac{n}{8} \le i \le \frac{n-4}{4} \end{cases}$$

Case (iv) Let n = 4t + 5, t = 0, 1, 2, 3, …

$$f(v_{4i+3}) = n - i - 1,\ 0 \le i \le \frac{n-5}{4}$$

Sub case (i) when t = 0, 2, 4, ….

$$f(v_n) = \frac{3n+1}{8};\ f(v_{n-1}) = n$$

$$f(v_{4i+1}) = i + 1,\ 0 \le i \le \frac{n-5}{4}$$

$$f(v_{4i-2}) = \frac{3n+1}{8} + i,\ 1 \le i \le \frac{n-1}{4}$$

$$f(v_{8i-4}) = \frac{n-1}{4} + i,\ 1 \le i \le \frac{n-5}{8},\ n > 5$$

$$f(v_{8i}) = n - \frac{n-1}{4} - i,\ 1 \le i \le \frac{n-5}{8},\ n > 5$$

Sub case (ii) when t = 1, 3, 5, ….

$$f(v_{4i+1}) = i + 1,\ 0 \le i \le \frac{n-1}{4}$$

$$f(v_{n-7}) = \frac{3n-3}{4};\ f(v_{n-3}) = n;$$

$$f(v_{8i-4}) = \frac{n+3}{4} + i,\ 1 \le i \le \frac{n-1}{8}$$

$$f(v_{8i}) = n - \frac{n+3}{4} - i,\ 1 \le i \le \frac{n-1}{8}$$

For n > 9

$$f(v_{4i-2}) = \frac{3n+5}{8} + i,\ 1 \le i \le \frac{n-9}{4}$$

If the vertices are labeled as above and if we define an induced function f^*: E(G) → {0, 1, 2, … n – 1} as $f^*(uv)$ = f(u)f(v) mod n then the sum of all edge labels is a multiple of n. Hence the path P_n, n > 1 is a modular multiplicative divisor graph.

Theorem: 3.2

The graph $P_{2,m}$ admits MMD Labeling.

Proof:

Let $P_{2,m}$ be the graph obtained by connecting two vertices u and v by means of "m" internally disjoint paths of length "2" each. The resulting graph is denoted by $P_{2,m}$.

Let V={u, v, $v_1, v_2, v_3, \ldots, v_m$} be the vertex set with |V| = m + 2 = n (say) and E = {$e_1, e_2, e_3, \ldots e_{2m}$} be the edge set of the graph $P_{2,m}$.

Define a bijection f : V(G)→{1,2,…,} and the induced function f^*: E(G) → {0, 1, 2, … n – 1} where $f^*(uv)$ = f(u) f(v) mod n as f(u) = 1, f(v) = n – 1, $f(v_m)$ = n and $f(v_i)$ = i + 1 for 1 ≤ i ≤ m – 1

Let the sum of all edge labels be S, then

$$S = \sum_{i=1}^{m} f^*(uv_i) + \sum_{i=1}^{m} f^*(vv_i)$$

We need to prove that S is a multiple of n.

$$S = [f(u)f(v_1) + f(u)f(v_2) + \ldots + f(u)f(v_m) + f(v)f(v_1) + f(v)f(v_2) + \ldots + f(v)f(v_m)] \bmod n$$

$$= [f(u) + f(v)] \sum_{i=1}^{m} f(v_i) \bmod n$$

$$= n \sum_{i=1}^{m} f(v_i) \bmod n$$

which is a multiple of n. Hence the graph $P_{2,m}$ admits modular multiplicative divisor labeling.

Example: 3.3

The MMD labeling of the graph obtained by connecting two vertices u and v by means of 5 internally disjoint paths of length 2 each is shown in Figure 1.

Theorem: 3.4

Shadow graph of a path $D_2(P_n)$ admits modular multiplicative divisor labeling.

Proof:

Let P_n be the path with n vertices and n – 1 edges and $D_2(P_n)$ be the shadow graph of a path P_n with $V = \{v_1', v_2', v_3', \ldots, v_n', v_1'', v_2'', v_3'', \ldots, v_n''\}$ as the vertex set and $E = E_1 \cup E_2 \cup E_3 \cup E_4$ as the edge set where

$$E_1 = \{v_i' v_{i+1}''\} \quad 1 \le i \le n-1$$

$$E_2 = \{v_{i+1}' v_i''\} \quad 1 \le i \le n-1$$

$$E_3 = \{v_i' v_{i+1}'\} \quad 1 \le i \le n-1$$

$$E_4 = \{v_i'' v_{i+1}''\} \quad 1 \le i \le n-1$$

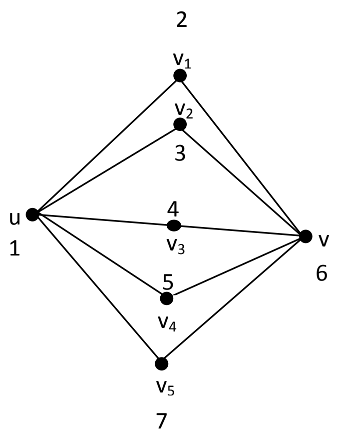

Figure 1. Modular Multiplicative Divisor Labeling For $P_{2,5}$.

Define a bijection f: $V(G) \to \{1, 2, \ldots, 2n\}$ and the induced function f^*: $E(G) \to \{0, 1, 2, \ldots 2n-1\}$ where $f^*(uv) = f(u)f(v) \bmod 2n$ as

$$f(v_n'') = 2n$$

$$f(v_i') = i \quad 1 \le i \le n$$

$$f(v_i'') = 2n - i \quad 1 \le i \le n-1$$

Let the sum of all edge labels be S, then

$$S = \sum_{i=1}^{n-1} [f^*(v_i' v_{i+1}'') + f^*(v_{i+1}' v_i'') + f^*(v_i' v_{i+1}') + f^*(v_i'' v_{i+1}'')]$$

We need to prove that S is a multiple of 2n.

$$S = \sum_{i=1}^{n-1} [f(v_i')f(v_{i+1}'') + f(v_{i+1}')f(v_i'') + f(v_i')f(v_{i+1}') + f(v_i'')f(v_{i+1}'')] \bmod 2n$$

$$= \left\{ \sum_{i=1}^{n-2} \left[i(2n-1-i) + (2n-i)(2n-1-i) \right] + \sum_{i=1}^{n-1} \left[(i+1)(2n-i) + i(i+1) \right] + (n-1)2n + 2n(n+1) \right\} \bmod 2n$$

which is a multiple of 2n.

Hence the shadow graph of a path $D_2(P_n)$ admits Modular Multiplicative Divisor labeling.

Example: 3.5

The MMD labeling for $D_2(P_6)$ is shown in Figure 2.

Theorem: 3.6

The Cartesian product $P_n \times P_1$, (n is not a multiple of 6) admits modular multiplicative divisor labeling.

Proof:

Case (i) when $n \equiv 1 \pmod 2$

Let $V = \{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ be the vertex set and $E = E_1 \cup E_2 \cup E_3$ be the edge set where $E_1 = \{v_i v_{i+1},$

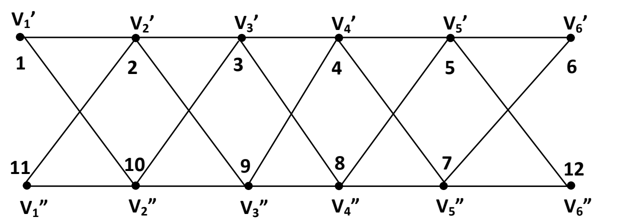

Figure 2. Modular multiplicative divisor labeling for $D_2(P_6)$.

$1 \le i < n\}$, $E_2 = \{v_i v_{n+i}, 1 \le i \le n\}$ and $E_3 = \{v_i v_{i+1}, n + 1 \le i \le 2n - 1\}$

Define a bijection $f: V \to \{1, 2, \ldots, 2n\}$ as $f(v_i) = i$, $1 \le i \le 2n$ and an induced function $f^* : E(G) \to \{0, 1, 2, \ldots, 2n - 1\}$ such that $f^*(v_i v_j) = f(v_i)f(v_j) \bmod 2n$ for all $v_i v_j \in E$ and $v_i, v_j \in V$ to prove that

$$\sum_{i=1}^{n-1} f^*(v_i v_{i+1}) + \sum_{i=1}^{n} f^*(v_i v_{n+i}) + \sum_{i=n+1}^{2n-1} f^*(v_i v_{i+1}) \text{ is a}$$

multiple of 2n.

For the edges in E_1:

$$\sum_{i=1}^{n-1} f^*(v_i v_{i+1}) = \sum_{i=1}^{n-1} f(v_i) f(v_{i+1}) \bmod 2n$$

$$= [n(n^2 - 1)/3] \bmod 2n$$

For the edges in E_2:

$$\sum_{i=1}^{n} f^*(v_i v_{n+i}) = \sum_{i=1}^{n} f(v_i) f(v_{n+i}) \bmod 2n$$

$$= [n(n + 1)(5n + 1)/6] \bmod 2n$$

For the edges in E_3:

$$\sum_{i=n+1}^{2n-1} f^*(v_i v_{i+1}) = \sum_{i=n+1}^{2n-1} f(v_i) f(v_{i+1}) \bmod 2n$$

$$= \{n(13n^2 - 12n - 1)/3\} \bmod 2n$$

Let S = Sum of all edge labels of $P_n \times P_1$, n is odd, then $S = [n(11n^2 - 6n - 1)/2] \bmod 2n$

When n is odd say 2k +1

$S = 2(2k + 1)(11k^2 + 8k + 1)$ which is a multiple of 2n.

Case (ii) when n is even and not a multiple of 6.

Let $V = \{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ be the vertex set and $E = E_1 \cup E_2$ be the edge set where $E_1 = \{v_i v_{i+1}, 1 \le i \le 2n - 1\}$ and $E_2 = \{v_i v_{2n+1-i}, 1 \le i \le n\}$.

Define a bijection $f : V \to \{1, 2, \ldots, 2n\}$ as $f(v_i) = i$, $1 \le i \le 2n$ and an induced function

$f^* : E(G) \to \{0, 1, 2, \ldots, 2n - 1\}$ such that $f^*(v_i v_j) = f(v_i) f(v_j) \bmod 2n$ for all $v_i v_j \in E$ and $v_i, v_j \in V$.

We need to prove that

$$\sum_{i=1}^{2n-1} f^*(v_i v_{i+1}) + \sum_{i=1}^{n-1} f^*(v_i v_{2n+1-i}) \text{ is a multiple of 2n.}$$

For the edges in E_1:

$$\sum_{i=1}^{2n-1} f^*(v_i v_{i+1}) = \sum_{i=1}^{2n-1} f(v_i) f(v_{i+1}) \bmod 2n$$

$$= [2n(4n^2 - 1)/3] \bmod 2n$$

For the edges in E_2:

$$\sum_{i=1}^{n-1} f^*(v_i v_{2n+1-i}) = \sum_{i=1}^{n-1} f(v_i) f(v_{2n+1-i}) \bmod 2n$$

$$= [2n(n^2 - 1)/3] \bmod 2n$$

Let S = Sum of all edge labels of $P_n \times P_1$, n is even and not a multiple of 6 say 3(2k + 1) ± 1. Then S is a multiple of 2n.

Example: 3.7

The MMD labeling for $P_7 \times P_1$, $P_8 \times P_1$ is shown in Figure 3 & 4.

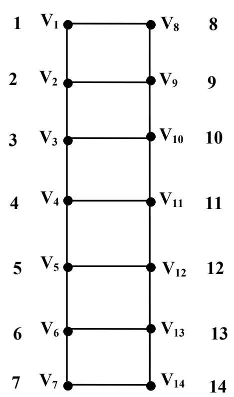

Figure 3. Modular multiplicative divisor labeling for $P_7 \times P_1$.

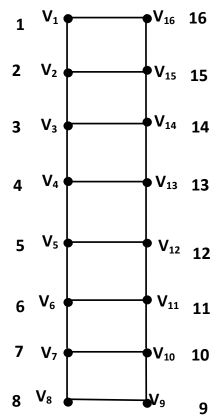

Figure 4. Modular multiplicative divisor labeling for $P_8 \times P_1$.

4. Upper Bound for Number of Edges of a Modular Multiplicative Divisor Graphs

In this section we study the number of edges in a modular multiplicative divisor graphs. Let $\lambda(n)$ be the maximum the number of edges in a modular multiplicative divisor graph of order n. We can find $\lambda(n)$ from the labeling of Complete graph K_n . Well-known result states that[6] maximum number of edges in a simple graph of order n is $n(n - 1)/2$. We proved that[3] Complete graph K_n, for all prime number $n > 3$ admits MMD labeling. It follows from the result that maximum the number of edges in a modular multiplicative divisor graph of order $n > 3$, a prime number is $n(n - 1)/2$. We put these facts into a theorem.

Theorem: 4.1

The maximum number of edges $\lambda(n)$ in a modular multiplicative divisor graph of order $n > 3$, a prime number is $n(n - 1)/2$.

Further we improve the above result by giving relaxation for the restriction to $n > 3$, a prime number. We[3] prove that the sum of all edge labels of a Complete graph K_n, for all prime number $n > 3$, is $n(n - 1)(n + 1)(3n + 2)/24$, which is unique. If we relax $n > 3$, a prime number to $n \equiv 1 \pmod 2$ as well as $n \equiv 1, 2 \pmod 3$, we have $(n - 1)(n + 1) \equiv 0 \pmod{24}$[8]. Hence we get modular multiplicative divisor graphs of order n, not only for prime numbers but also for composite numbers like 5^2, 7^2, 11^2, 13^2, …, 5^3, 7^3, 11^3, 13^3, … (i.e $n^m \equiv 1 \pmod{24}$, $m \in N - \{1\}$) and 35, 55, 65, 77,… admits MMD labeling. We conclude that the maximum number of edges $\lambda(n)$ in a modular multiplicative divisor graph of order $n > 3$, $n \equiv 1 \pmod 2$ and $n \equiv 1, 2 \pmod 3$ is $n(n - 1)/2$. From this study we get the following theorem.

Theorem: 4.2

The maximum number of edges $\lambda(n)$ in a modular multiplicative divisor graph of order $n > 3$, $n \equiv 1 \pmod 2$ and $n \equiv 1, 2 \pmod 3$ is $n(n - 1)/2$.

5. Conclusion

We prove that some path related graphs admits modular multiplicative divisor labeling and also studied the upper bound for the number of edges in a modular multiplicative divisor graphs of order $n > 3$, $n \equiv 1 \pmod 2$ and $n \equiv 1, 2 \pmod 3$. We investigated the upper bound for number of edges $\lambda(n)$ in a modular multiplicative divisor graph of order $n > 3$, for $n \equiv 0 \pmod 2$ and $n \equiv 3 \pmod 6$.

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