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Interval Valued Intuitionistic $(\bar{S}, \bar{T})$- Fuzzy Ideals of Ternary Semigroups

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Abstract

In this paper, the concept of interval valued intuitionistic fuzzy ternary subsemigroup (ideal) of a ternary semigroup with respect to interval t-norm $\bar{T}$ and interval t-conorm $\bar{S}$ is given and the characteristic properties are described. We characterized some other classes of ternary semigroups by the properties these interval valued intuitionistic fuzzy ternary subsemigroup (ideal) of a ternary semigroup. The homomorphic image and inverse image are also investigated.

Keywords: Ternary Semigroups, Interval Valued Intuitionistic $(\bar{S}, \bar{T})$- Fuzzy Ternary Subsemigroups (Ideals).

1. Introduction

In 1932, Lehmer introduced the concept of ternary semigroup [1]. The algebraic structures of ternary semigroups were also studied by some authors, for example, Sioson studied ideals in ternary semigroups [2]. Dixit and Dewan studied quasi-ideals and bi-ideals in ternary semigroups [3], Iampan studied minimal and maximal lateral ideals of ternary semigroups [4].

The concept of a fuzzy set was formulated by Zadeh in [5], since then, the theory of fuzzy sets developed by Zadeh and others has evoked tremendous interest among researchers working in different branches of mathematics. Fuzzy semigroups were introduced by Kuroki [6] as a generalization of classical semigroups. Many classes of semigroups were studied by Kuroki using fuzzy ideals in [7]. Mordeson et. al. [8] gave a systematic exposition of fuzzy semigroups, where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. Kar and Sarkar [9], introduced fuzzy ideals of ternary semigroups, also see [10–24].

After the introduction of the concept of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets [25, 26] and interval valued intuitionistic fuzzy sets [27, 28] are among them. Kim and Jun [29] introduced the concept of intuitionistic fuzzy ideals of semigroups and in [30], Kim and Lee studied intuitionistic fuzzy bi-ideals of semigroups.

Kim et. al. [31] gave the concept of intuitionistic (T, S) normed fuzzy ideals of Γ-rings. Gujin and Xiapping [32], introduced the concept of interval-valued fuzzy subgroups induced by T-triangular norms. Akram and Dar in [33] introduced the idea of fuzzy left h-ideal in hemirings with respect to an s-norm. Zhan [34], studied the fuzzy left h-ideals in hemirings with t-norms. There are several authors who applied the theory of intuitionistic (S, T)-fuzzy sets to different algebraic structures for instance, Akram [35], Aygunoglu et al. [36], Davvaz et al.

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[37–39], Dudek et al. [40], Hedayati [41–43], Lee and Kim [44], Shum and Akram [45], and Zhan et al. [46, 47].

In this paper, we studied the idea of interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ideals in ternary semigroups and investigated some of related properties.

2. Preliminaries

Throughout this paper G will denote a ternary semigroup.

Definition 2.1: *[1] A ternary semigroup is an algebraic structure (G, f) such that G is a non-empty set and f: $G^3 \rightarrow G$ is a ternary operation satisfying the following associative law:*

$$f(f(a,b,c),d,e) = f(a, f(b,c,d),e) = f(a,b, f(c,d,e)).$$

For simplicity we write $f(a, b, c)$ as abc.

Definition 2.2: *[2] A non-empty subset A of a ternary semigroup G is said to be a ternary subsemigroup of G if $AAA = A^3 \subseteq A$ that is $abc \in A$ for all a, b, $c \in A$.*

Definition 2.3: *[2] By a left (right, lateral) ideal of a ternary semigroup G we mean a non-empty subset A of G such that $GGA \subseteq A$ ($AGG \subseteq A$, $GAG \subseteq A$). By a two sided ideal, we mean a subset A of G which is both a left and a right ideal of G. If a non-empty subset of G is a left, right and lateral ideal of G, then it is called an ideal of G.*

Definition 2.4: *[9] An element a of a ternary semigroup G is called regular if there exists an element $x \in G$ such that $axa = a$. A ternary semigroup G is called regular if every element of G is regular.*

Throughout this paper, let I be a closed unit interval, i.e., $I = [0,1]$. An interval number is $\bar{a} = [a^-, a^+]$, where $0 \le a^- \le a^+ \le 1$. Let $[I]$ denote the set of all interval numbers, i.e., $[I] = \{\bar{a} = [a^-, a^+] : a^- \le a^+\}$, where $a^-, a^+ \in I$ and the elements in $[I]$ are called the interval numbers on I.

We define the operation of the supremum, infimum and the orders with respect to the interval numbers on $[I]$ as follows:

Let $\bar{a}_j \in [I]$, where $\bar{a}_j = [a_j^-, a_j^+]$, $a_j^-, a_j^+ \in I$, for all $j \in J$, J be an index set. Define

$$\underset{j\in J}{\wedge} a_j^- = \inf\{a_j^- \ : \ j \in J\}$$

$$\text{and} \quad \underset{j\in J}{\vee} a_j^- = \sup\{a_j^- \ : \ j \in J\},$$

$$\inf \bar{a}_j = [\wedge a_j^-, \wedge a_j^+], \sup \bar{a}_j = [\vee a_j^-, \vee a_j^+].$$

In particular, wherever $\bar{a}, \bar{b} \in [I]$, $\bar{a} = [a^-, a^+]$, $b = [b^-, b^+]$, we define

(1) $\bar{a} \le \bar{b}$ iff $a^- \le b^-, a^+ \le b^+$;
(2) $\bar{a} = \bar{b}$ iff $a^- = b^-, a^+ = b^+$;
(3) $\bar{a} < \bar{b}$ iff $\bar{a} \le \bar{b}$ and $\bar{a} \ne \bar{b}$.

From above, it is easy to infer that $\bar{a} < \bar{b}$ iff $a^- < b^-$ and $a^+ \le b^+$ or $a^- \le b^-$ and $a^+ < b^+$.

Obviously, $([I], \le, \sup, \inf)$ constitutes a complete lattice with the least element $\bar{0} = [0, 0]$ and the greatest element $\bar{1} = [1,1]$ (see [48]).

Definition 2.5: *[49] Let X be an ordinary set, then the mapping $\bar{A} : X \rightarrow [I]$ is called an interval-valued fuzzy set (for short, IVFS) on X.*

Let $IF(X)$ denote the family of the interval-valued fuzzy sets on X. For each $\bar{A} \in IF(X)$, we have $\bar{A}(x) = [A^-(x), A^+(x)]$, where $A^-(x) \le A^+(x)$, for all $x \in X$. Then the ordinary fuzzy set $A^- : X \rightarrow I$ and $A^+ : X \rightarrow I$ is called a lower fuzzy set and an upper fuzzy set of $\bar{A}$, respectively. In addition, we define $\phi(x) = [0,0], X(x) = [1,1]$, for all $x \in X$. Obviously $\phi, X \in IF(X)$.

Let $\bar{A}, \bar{B} \in IF(X), x \in X$, we define

$$(\bar{A} \cup \bar{B})(x) = \bar{A}(x) \vee \bar{B}(x)$$
$$\text{and} \quad (\bar{A} \cap \bar{B})(x) = \bar{A}(x) \wedge \bar{B}(x).$$

Let $[I] \times [I]$ denote set of all double interval numbers, i.e., $[I] \times [I] = \{(\bar{a}, \bar{b}) = ([a^-, a^+], [b^-, b^+]) : a^- \le a^+, b^- \le b^+, a^+ + b^+ \le 1\}$ where the elements in $[I] \times [I]$ are called the double interval numbers on $I \times I$. We define the operation of the supremum, infimum and the orders with respect to the double interval numbers on $[I] \times [I]$, as follows:

Let $(\bar{a}_i, \bar{b}_i) \in [I] \times [I]$ where $\bar{a}_i = [a_i^-, a_i^+]$, $\bar{b}_i = [b_i^-, b_i^+]$ with $a_i^+ + b_i^+ \le 1$, for all $i \in J, J$ be an index set. Define

$$\underset{j\in J}{\wedge}(\bar{a}_i, \bar{b}_i) = \left(\underset{j\in J}{\wedge}\bar{a}_i, \underset{j\in J}{\wedge}\bar{b}_i\right) = \left(\left[\underset{j\in J}{\wedge}a_i^-, \underset{j\in J}{\wedge}a_i^+\right], \left[\underset{j\in J}{\wedge}b_i^-, \underset{j\in J}{\wedge}b_i^+\right]\right),$$

$$\underset{j\in J}{\vee}(\bar{a}_i, \bar{b}_i) = \left(\underset{j\in J}{\vee}\bar{a}_i, \underset{j\in J}{\vee}\bar{b}_i\right) = \left(\left[\underset{j\in J}{\vee}a_i^-, \underset{j\in J}{\vee}a_i^+\right], \left[\underset{j\in J}{\vee}b_i^-, \underset{j\in J}{\vee}b_i^+\right]\right).$$

Obviously, $([I] \times [I], \le, \sup, \inf)$ constitutes a complete lattice with the least element $\tilde{0} = ([0,0],[1,1])$ and the greatest element $\tilde{1} = ([1,1],[0,0])$.

Definition 2.6: *[28] Let X be an ordinary set, then the mapping $\tilde{A} : X \to [I]\times[I]$ defined by $\tilde{A}(x) = [\bar{M}_A(x), \bar{N}_A(x)]$, where $M_A^+(x) + N_A^+(x) \le 1$, for all $x \in X$ is called an interval-valued intuitionistic fuzzy set (for short, IIF-set) on X.*

Then the interval-valued fuzzy sets $\bar{M}_A : X \to [I]$, define by $\bar{M}_A(x) = \left[M_A^-(x), M_A^+(x)\right]$, where $M_A^-(x) \le M_A^+(x)$ and $\bar{N}_A : X \to [I]$, defined by $\bar{N}_A(x) = [N_A^-(x), N_A^+(x)]$, where $N_A^-(x) \le N_A^+(x)$, denote the degree of belongingness and the degree of nonbelongingness of each element $x \in X$ to the set $\tilde{A}$, respectively.

Definition 2.7: *[28] Let $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ and $\tilde{B} = \left(\bar{M}_B, \bar{N}_B\right)$ be interval-valued intuitionistic fuzzy sets in a set S. Then*

(1) $\tilde{A} \subseteq \tilde{B}$ if and only if $\bar{M}_A \le \bar{M}_B$ and $\bar{N}_A \ge \bar{N}_B$.
(2) $\tilde{A}^c = \left(\bar{N}_A, \bar{M}_A\right)$.
(3) $\tilde{A} \cap \tilde{B} = \left(\bar{M}_A \wedge \bar{M}_B, \bar{N}_A \vee \bar{N}_B\right)$.
(4) $\tilde{A} \cup \tilde{B} = \left(\bar{M}_A \vee \bar{M}_B, \bar{N}_A \wedge \bar{N}_B\right)$.
(5) $\Box\, \tilde{A} = \left(\bar{M}_A, \bar{M}_A^*\right)$ where $\bar{M}_A^* = 1 - \bar{M}_A$.
(6) $\Diamond\, \tilde{A} = \left(\bar{N}_A^*, \bar{N}_A\right)$ where $\bar{N}_A^* = 1 - \bar{N}_A$.

Let IIF(X) denote the family of all the interval-valued intuitionistic fuzzy sets on *X*.

Definition 2.8: *[50] Let $\tilde{A}$ and $\tilde{B}$ be interval-valued intuitionistic fuzzy sets. Then define*

(a) $\tilde{A} \subset \tilde{B}$ if and only if $\bar{M}_A \le \bar{M}_B$ and $\bar{N}_A \ge \bar{N}_B$.
(b) $\tilde{A} = \tilde{B}$ if and only if $\tilde{A} \subset \tilde{B}$ and $\tilde{B} \subset \tilde{A}$.
(c) If $\{A_i : i \in J\}$ is a family of interval-valued intuitionistic fuzzy sets, then

$$\bigcup_{i\in J}\tilde{A}_i(x) = \left(\cup \bar{M}_{A_i}(x), \cap \bar{N}_{A_i}(x)\right) = \begin{pmatrix}[\vee M_{A_i}^-(x), \vee M_{A_i}^+(x)],\\ [\wedge N_{A_i}^-(x), \wedge N_{A_i}^+(x)]\end{pmatrix},$$

and

$$\bigcap_{i\in J}\tilde{A}_i(x) = \left(\bigcap \bar{M}_{A_i}(x), \bigcup \bar{N}_{A_i}(x)\right) = \begin{pmatrix}[\wedge M_{A_i}^-(x), \wedge M_{A_i}^+(x)],\\ [\vee N_{A_i}^-(x), \vee N_{A_i}^+(x)]\end{pmatrix}.$$

(d) $\tilde{0} = ([0,0],[1,1])$ and $\tilde{1} = ([1,1],[0,0])$.

Definition 2.9: *Let $\tilde{A}$ be an interval-valued intuitionistic fuzzy set. For arbitrary $[\lambda_1, \lambda_2], [\theta_1, \theta_2] \in [I]$ with $\lambda_2 + \theta_2 \le 1$, let the set $\tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])} = \{x \in X : \bar{M}_A(x) \ge [\lambda_1, \lambda_2],\ \bar{N}_A(x) \le [\theta_1, \theta_2]\}$, then $\tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])}$ is called a $([\lambda_1,\lambda_2],[\theta_1,\theta_2])$-level set of $\tilde{A}$. Obviously, $\tilde{A}_{([\lambda_1,\lambda_2],[\theta 1,\theta 2])} = \bar{M}_{A_{[\lambda_1,\lambda_2]}} \cap \bar{N}_{A_{[\theta 1,\theta 2]}}$, where $\bar{M}_{A_{[\lambda_1,\lambda_2]}} = M_{A_{\lambda_1}}^- \cap M_{A_{\lambda_2}}^+$ and $\bar{N}_{A_{[\theta_1,\theta_2]}} = N_{A_{\theta_1}}^- \cap N_{A_{\theta_2}}^+$.*

Definition 2.10: *[36] A mapping $\bar{T} : [I]\times[I] \to [I]$ is called an interval t-norm defined on $[I]\times[I]$, if the following conditions are satisfied:*

(1) $\bar{T}(\bar{a}, \bar{1}) = \bar{a},\ \forall a \in [I]$,
(2) $\bar{T}(\bar{a}, \bar{b}) = \bar{T}(\bar{b}, \bar{a}),\ \forall \bar{a}, \bar{b} \in [I]$,
(3) $\bar{T}(\bar{a}, \bar{T}(\bar{b}, \bar{c})) = \bar{T}(\bar{T}(\bar{a}, \bar{b}), \bar{c}),\ \forall \bar{a}, \bar{b}, \bar{c} \in [I]$,
(4) If $\bar{a} \le \bar{c}$, $\bar{b} \le \bar{d}$, then $\bar{T}(\bar{a}, \bar{b}) \le \bar{T}(\bar{c}, \bar{d}),\ \forall \bar{a}, \bar{b}, \bar{c}, \bar{d} \in [I]$.

If $\bar{T}(\bar{a}, \bar{a}) = \bar{a}$ for all $\bar{a} \in [I]$, then $\bar{T}$ is called an idempotent interval t-norm.

Proposition 2.11: *[36] Let $\bar{T}$ be an interval t-norm. Then the following conditions are satisfied:*

(i) $\bar{T}(\bar{a}, \bar{0}) = \bar{0},\ \forall \bar{a} \in [I]$.
(ii) $\bar{T}(\bar{a}, \bar{b}) \le \bar{a} \wedge \bar{b},\ \forall \bar{a}, \bar{b} \in [I]$.

Definition 2.12: *[36] A mapping $\bar{S} : [I]\times[I] \to [I]$ is called an interval t-conorm defined on $[I]\times[I]$, if the following conditions are satisfied:*

(1) $\bar{S}(\bar{a}, \bar{0}) = \bar{a},\ \forall \bar{a} \in [I]$,
(2) $\bar{S}(\bar{a}, \bar{b}) = \bar{S}(\bar{b}, \bar{a}),\ \forall \bar{a}, \bar{b} \in [I]$,
(3) $\bar{S}(\bar{a}, \bar{S}(\bar{b}, \bar{c})) = \bar{S}(\bar{S}(\bar{a}, \bar{b}), \bar{c}),\ \forall \bar{a}, \bar{b}, \bar{c} \in [I]$,
(4) If $\bar{a} \le \bar{c}, \bar{b} \le \bar{d}$, then $\bar{S}(\bar{a}, \bar{b}) \le \bar{S}(\bar{c}, d),\ \forall \bar{a}, \bar{b}, \bar{c}, \bar{d} \in [I]$.

If $\bar{S}(\bar{a}, \bar{a}) = \bar{a}$ for all $\bar{a} \in [I]$, then $\bar{S}$ is called an idempotent interval t-conorm.

Proposition 2.13: *[36] Let $\bar{S}$ be an interval t-conorm. Then the following conditions are satisfied:*

(i) $\bar{S}(\bar{a}, \bar{1}) = \bar{1},\ \forall \bar{a} \in [I]$.
(ii) $\bar{S}(\bar{a}, \bar{b}) \ge \bar{a} \vee \bar{b},\ \forall \bar{a}, \bar{b} \in [I]$.

3. Interval Valued Intuitionistic ($\bar{S}$, $\bar{T}$)-Fuzzy Ideals of Ternary Semigroups

In this section, we define interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy left (right, lateral) ideals of ternary semigroups and prove some basic properties of these ideals.

Definition 3.1: *Let G be a ternary semigroup. An IIF-subset $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ of G is called an interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ternary subsemigroup of G if*

$$\bar{M}_A(xyz) \geq \bar{T}(\bar{M}_A(x), \bar{M}_A(y), \bar{M}_A(z))$$
$$\text{and } \bar{N}_A(xyz) \leq \bar{S}(\bar{N}_A(x), \bar{N}_A(y), \bar{N}_A(z))$$

for all $x, y, z \in G$.

Definition 3.2: *An IIF-subset $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ of G is called an interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy left (right, lateral) ideal of G if $\bar{M}_A(xyz) \geq \bar{M}_A(z)$ ($\bar{M}_A(xyz) \geq \bar{M}_A(x)$, $\bar{M}_A(xyz) \geq \bar{M}_A(y)$) and $\bar{N}_A(xyz) \leq \bar{N}_A(z)$ ($\bar{N}_A(xyz) \leq \bar{N}_A(x)$, $\bar{N}_A(xyz) \leq \bar{N}_A(y)$) for all $x, y, z \in G$.*

An IIF-subset $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ of G is called an interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ideal of G if it is an interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy left ideal, interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy right ideal and interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy lateral ideal of G.

Example 3.3: Let $G = \{1, 2, 3, 4\}$ be a ternary semigroup with the following Cayley table

·	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	4	1	1
4	1	3	1	1

Let $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ be an interval valued intuitionistic fuzzy subset of G such that

$$\tilde{A} = \left(\begin{array}{l} \left(\frac{1}{[0.65, 0.7]}, \frac{2}{[0.5, 0.57]}, \frac{3}{[0.56, 0.59]}, \frac{4}{[0.56, 0.59]} \right), \\ \left(\frac{1}{[0.2, 0.28]}, \frac{2}{[0.35, 0.4]}, \frac{3}{[0.35, 0.4]}, \frac{4}{[0.3, 0.36]} \right) \end{array} \right)$$

Corresponding interval t-norm and interval s-norm are defined as

$$\bar{T}(\bar{x}, \bar{y}) = [\max\{x^- + y^- - 1, 0\}, \max\{x^+ + y^+ - 1, 0\}]$$
$$\text{and } \bar{S}(\bar{x}, \bar{y}) = [\min\{x^- + y^-, 1\}, \min\{x^+ + y^+, 1\}]$$

for all $\bar{x}, \bar{y} \in [I]$. By routine calculations we can check that the IIF-subset $\tilde{A}$ is an interval valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ideal of G.

Definition 3.4: *Let $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$, $\tilde{B} = \left(\bar{M}_B, \bar{N}_B\right)$ and $\tilde{C} = \left(\bar{M}_C, \bar{N}_C\right)$ be three IIF-subsets of G. The product $\tilde{A} \circ \tilde{B} \circ \tilde{C} = (\bar{M}_{A \circ B \circ C}, \bar{N}_{A \circ B \circ C})$ is defined by*

$$\bar{M}_{A \circ B \circ C}(x) = \begin{cases} \bigvee_{x = abc} \bar{T}\left(\bar{M}_A(a), \bar{M}_B(b), \bar{M}_C(c)\right), \\ \text{if } \exists\, a, b, c \in G, \text{ such that } x = abc \\ [0, 0], \ \textit{otherwise}. \end{cases}$$

$$\bar{N}_{A \circ B \circ C}(x) = \begin{cases} \bigwedge_{x = abc} \bar{S}\left(\bar{N}_A(a), \bar{N}_B(b), \bar{N}_C(c)\right), \\ \text{if } \exists\, a, b, c \in G, \text{ such that } x = abc \\ [1, 1], \ \textit{otherwise}. \end{cases}$$

We note that the ternary semigroup G can be considered as an IIF-subset of itself and we write $\tilde{G} = \left(\bar{M}_G, \bar{N}_G\right)$. And $\tilde{G} = \left(\bar{M}_G, \bar{N}_G\right)$ will be carried out in operations with an IIF-subset $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ such that $\bar{M}_G$ and $\bar{N}_G$ will be used in collaboration with $\bar{M}_A$ and $\bar{N}_A$ respectively.

We shall denote the set of all interval-valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy subsets of G by $IIF(G, S, T)$.

Lemma 3.5: *Let $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ and $\tilde{B} = \left(\bar{M}_B, \bar{N}_B\right)$ be two interval-valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ternary subsemigroups of G. Then $(\bar{T}(\bar{M}_A, \bar{M}_B), \bar{S}(\bar{N}_A, \bar{N}_B))$ is also an interval-valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ternary subsemigroup of G.*

Proof: Let $\tilde{A} = \left(\bar{M}_A, \bar{N}_A\right)$ and $\tilde{B} = \left(\bar{M}_B, \bar{N}_B\right)$ be two interval-valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ternary subsemigroups of G. Then for $x, y, z \in G$, we have

$$\begin{aligned} \bar{T}(\bar{M}_A, \bar{M}_B)(xyz) &= \bar{T}(\bar{M}_A(xyz), \bar{M}_B(xyz)) \\ &\geq \bar{T}\left(\begin{array}{l} \bar{T}(\bar{M}_A(x), \bar{M}_A(y), \bar{M}_A(z)), \\ \bar{T}(\bar{M}_B(x), \bar{M}_B(y), \bar{M}_B(z)) \end{array} \right) \\ &= \bar{T}\left(\begin{array}{l} \bar{T}(\bar{M}_A(x), \bar{M}_B(x)), \\ \bar{T}(\bar{M}_A(y), \bar{M}_B(y)), \\ \bar{T}(\bar{M}_A(z), \bar{M}_B(z)) \end{array} \right) \\ &= \bar{T}\left(\begin{array}{l} \bar{T}(\bar{M}_A, \bar{M}_B)(x), \\ \bar{T}(\bar{M}_A, \bar{M}_B)(y), \\ \bar{T}(\bar{M}_A, \bar{M}_B)(z) \end{array} \right). \end{aligned}$$

And

$$\begin{aligned} \bar{S}(\bar{N}_A, \bar{N}_B)(xyz) &= \bar{S}(\bar{N}_A(xyz), \bar{N}_B(xyz)) \\ &\leq \bar{S}\left(\begin{array}{l} \bar{S}(\bar{N}_A(x), \bar{N}_A(y), \bar{N}_A(z)), \\ \bar{S}(\bar{N}_B(x), \bar{N}_B(y), \bar{N}_B(z)) \end{array} \right) \\ &= \bar{S}\left(\begin{array}{l} \bar{S}(\bar{N}_A(x), \bar{N}_B(x)), \\ \bar{S}(\bar{N}_A(y), \bar{N}_B(y)), \\ \bar{S}(\bar{N}_A(z), \bar{N}_B(z)) \end{array} \right) \\ &= \bar{S}\left(\begin{array}{l} \bar{S}(\bar{N}_A, \bar{N}_B)(x), \\ \bar{S}(\bar{N}_A, \bar{N}_B)(y), \\ \bar{S}(\bar{N}_A, \bar{N}_B)(z) \end{array} \right). \end{aligned}$$

Hence this shows that $(\bar{T}(\bar{M}_A, \bar{M}_B), \bar{S}(\bar{N}_A, \bar{N}_B))$ is an interval-valued intuitionistic $(\bar{S}, \bar{T})$-fuzzy ternary sub-semi-group of G.

Lemma 3.6: *If $\tilde{A}=(\bar{M}_A,\bar{N}_A)$ and $\tilde{B}=(\bar{M}_B,\bar{N}_B)$ are two interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideals of G. Then $(\bar{T}(\bar{M}_A,\bar{M}_B),\bar{S}(\bar{N}_A,\bar{N}_B))$ is also an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) of G.*

Lemma 3.7: *If $\{\tilde{A}_i\}_{i\in I}$ is a family of interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroups of G. Then,*

(1) $\bigcap_{i\in I}\tilde{A}_i$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G,
(2) $\bigcup_{i\in I}\tilde{A}_i$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G,

where $\bigcap_{i\in I}\tilde{A}_i=\left(\wedge_{i\in I}\bar{M}_{A_i},\vee_{i\in I}\bar{N}_{A_i}\right)$ and $\bigcup_{i\in I}\tilde{A}_i=\left(\vee_{i\in I}\bar{M}_{A_i},\wedge_{i\in I}\bar{N}_{A_i}\right)$.

Proof: The proof is straightforward.

Lemma 3.8: *If $\{\tilde{A}_i\}_{i\in I}$ is a family interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideals of G. Then,*

(1) $\bigcap_{i\in I}\tilde{A}_i$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideal of G,
(2) $\bigcup_{i\in I}\tilde{A}_i$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideal of G,

where $\bigcap_{i\in I}\tilde{A}_i=\left(\wedge_{i\in I}\bar{M}_{A_i},\vee_{i\in I}\bar{N}_{A_i}\right)$ and $\bigcup_{i\in I}\tilde{A}_i=\left(\vee_{i\in I}\bar{M}_{A_i},\wedge_{i\in I}\bar{N}_{A_i}\right)$.

Proof: The proof is straightforward.

Theorem 3.9: *The product of three interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideals of G is again an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideal of G.*

Proof: The proof is straightforward.

Theorem 3.10: *Let $\tilde{A}$ be an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy right ideal, $\tilde{B}$ an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy lateral ideal and $\tilde{C}$ an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of a ternary semigroup G. Then*

$$\bar{M}_{A\circ B\circ C}\le\bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)\text{ and }\bar{N}_{A\circ B\circ C}\ge\bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C).$$

Proof: Let $\tilde{A}$ be an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy right ideal, $\tilde{B}$ an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy lateral ideal and $\tilde{C}$ an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of a ternary semigroup G. If x is not expressible as $x=abc$, then $\bar{M}_{A\circ B\circ C}(x)=[0,0]\le\bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)$ and $\bar{N}_{A\circ B\circ C}(x)=[1,1]\ge\bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)(x)$.

If x is expressible as $x=abc$, then

$$\begin{aligned}\bar{M}_{A\circ B\circ C}(x)&=\vee_{x=abc}\bar{T}(\bar{M}_A(a),\bar{M}_B(b),\bar{M}_C(c))\\&\le\vee_{x=abc}\bar{T}(\bar{M}_A(abc),\bar{M}_B(abc),\bar{M}_C(abc))\\&=\vee_{x=abc}\bar{T}(\bar{M}_A(x),\bar{M}_B(x),\bar{M}_C(x))\\&=\bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)(x).\end{aligned}$$

And

$$\begin{aligned}\bar{N}_{A\circ B\circ C}(x)&=\wedge_{x=abc}\bar{S}(\bar{N}_A(a),\bar{N}_B(b),\bar{N}_C(c))\\&\ge\wedge_{x=abc}\bar{S}(\bar{N}_A(abc),\bar{N}_B(abc),\bar{N}_C(abc))\\&=\wedge_{x=abc}\bar{S}(\bar{N}_A(x),\bar{N}_B(x),\bar{N}_C(x))\\&=\bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)(x).\end{aligned}$$

Thus $\bar{M}_{A\circ B\circ C}\le\bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)$ and $\bar{N}_{A\circ B\circ C}\ge\bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)$.

Theorem 3.11: *Let $\tilde{A}$ be an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy right ideal and $\tilde{B}$ be an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of a ternary semigroup G. Then*

$$\bar{M}_{A\circ G\circ B}\le\bar{T}(\bar{M}_A,\bar{M}_B)\text{ and }\bar{N}_{A\circ G\circ B}\ge\bar{S}(\bar{N}_A,\bar{N}_B).$$

Where $\bar{M}_G(x)=[1,1]$ and $\bar{N}_G(x)=[0,0]$ for all x in G.

Proof: Let $\tilde{A}$ be an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy right ideal and $\tilde{B}$ be an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of G. Let $a\in G$. If a is not expressible as $a=xyz$ for some $x,y,z\in G$, then

$$\bar{M}_{A\circ G\circ B}(a)=[1,1]\ge\bar{T}(\bar{M}_A,\bar{M}_B)(a).$$

and,

$$\bar{N}_{A\circ G\circ B}(a)=[0,0]\le\bar{S}(\bar{N}_A,\bar{N}_B)(a).$$

If a is expressible as $a=xyz$ for some $x,y,z\in G$, then

$$\begin{aligned}\bar{M}_{A\circ G\circ B}(a)&=\vee_{a=xyz}\bar{T}(\bar{M}_A(x),\bar{M}_G(y),\bar{M}_B(z))\\&=\vee_{a=xyz}\bar{T}(\bar{M}_A(x),[1,1],\bar{M}_B(z))\\&=\vee_{a=xyz}\bar{T}(\bar{M}_A(x),\bar{M}_B(z))\\&\le\vee_{a=xyz}\bar{T}(\bar{M}_A(xyz),\bar{M}_B(xyz))\\&=\vee_{a=xyz}\bar{T}(\bar{M}_A(a),\bar{M}_B(a))\\&=\bar{T}(\bar{M}_A,\bar{M}_B)(a).\end{aligned}$$

and,

$$\bar{N}_{A\circ G\circ B}(a)=\wedge_{a=xyz}\bar{S}(\bar{N}_A(x),\bar{N}_G(y),\bar{N}_B(z))$$

$$\begin{aligned}&= \bigwedge_{a=xyz} \bar{S}(\bar{N}_A(x),[0,0],\bar{N}_B(z))\\&= \bigwedge_{a=xyz} \bar{S}(\bar{N}_A(x),\bar{N}_B(z))\\&\geq \bigwedge_{a=xyz} \bar{S}(\bar{N}_A(xyz),\bar{N}_B(xyz))\\&= \bigwedge_{a=xyz} \bar{S}(\bar{N}_A(a),\bar{N}_B(a))\\&= \bar{S}(\bar{N}_A,\bar{N}_B)(a).\end{aligned}$$

Thus $\bar{M}_{A\circ G\circ B} \leq \bar{T}(\bar{M}_A,\bar{M}_B)$ and $\bar{N}_{A\circ G\circ B} \geq \bar{S}(\bar{N}_A,\bar{N}_B)$.

Theorem 3.12: *Let $\tilde{A} \in IIF(G,S,T)$. Then $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G if and only if*

$$\tilde{A}\circ\tilde{A}\circ\tilde{A}\subseteq\tilde{A}.$$

Proof: The proof is straightforward.

Theorem 3.13: *Let $\tilde{A} \in IIF(G,S,T)$. Then $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideal of G if and only if $\tilde{G}\circ\tilde{G}\circ\tilde{A}\subseteq\tilde{A}$ ($\tilde{A}\circ\tilde{G}\circ\tilde{G}\subseteq\tilde{A}$, $\tilde{G}\circ\tilde{A}\circ\tilde{G}\subseteq\tilde{A}$).*

Proof: Suppose $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of G. To show that $\tilde{G}\circ\tilde{G}\circ\tilde{A}\subseteq\tilde{A}$. Let $x\in G$, such that

$$\begin{aligned}\bar{M}_{G\circ G\circ A}(x) &= \bigvee_{x=abc} \bar{T}\left(\bar{M}_G(a),\bar{M}_G(b),\bar{M}_A(c)\right)\\&= \bigvee_{x=abc} \bar{T}\left([1,1],[1,1],\bar{M}_A(c)\right)\\&= \bigvee_{x=abc} \bar{M}_A(c) \leq \bigvee_{x=abc} \bar{M}_A(abc)\\&= \bar{M}_A(x).\end{aligned}$$

And

$$\begin{aligned}\bar{N}_{G\circ G\circ A}(x) &= \bigwedge_{x=abc} \bar{S}\left(\bar{N}_G(a),\bar{N}_G(b),\bar{N}_A(c)\right)\\&= \bigwedge_{x=abc} \bar{S}\left([0,0],[0,0],\bar{N}_A(c)\right)\\&= \bigwedge_{x=abc} \bar{N}_A(c) \geq \bigwedge_{x=abc} \bar{N}_A(abc)\\&= \bar{N}_A(x).\end{aligned}$$

If x is not expressible as $x=abc$ for all $a,b,c\in G$, then

$$\bar{M}_{G\circ G\circ A}(x) = [0,0] \leq \bar{M}_A(x)$$
$$\text{and } \bar{N}_{G\circ G\circ A}(x) = [1,1] \geq \bar{N}_A(x).$$

Hence $\tilde{G}\circ\tilde{G}\circ\tilde{A}\subseteq\tilde{A}$.

Conversely, assume that $\tilde{G}\circ\tilde{G}\circ\tilde{A}\subseteq\tilde{A}$. Let $x,y,z\in S$. Then

$$\begin{aligned}\bar{M}_A(xyz) \geq \bar{M}_{G\circ G\circ A}(xyz)&\\&= \bigvee_{xyz=abc} \bar{T}\left(\bar{M}_G(a),\bar{M}_G(b),\bar{M}_A(c)\right)\\&= \bigvee_{xyz=abc} \bar{T}\left([1,1],[1,1],\bar{M}_A(c)\right)\\&= \bigvee_{xyz=abc} \bar{M}_A(c) \geq \bar{M}_A(z).\end{aligned}$$

And

$$\begin{aligned}\bar{N}_A(xyz) \geq \bar{N}_{G\circ G\circ A}(xyz)&\\&= \bigwedge_{xyz=abc} \bar{S}\left(\bar{N}_G(a),\bar{N}_G(b),\bar{N}_A(c)\right)\\&= \bigwedge_{xyz=abc} \bar{S}\left([0,0],[0,0],\bar{N}_A(c)\right)\\&= \bigwedge_{xyz=abc} \bar{N}_A(c) \geq \bar{N}_A(z).\end{aligned}$$

Thus $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of G. The other cases can be seen in a similar way.

Theorem 3.14: *Let $\tilde{A}=\left(\bar{M}_A,\bar{N}_A\right)$ be an IIF-subset of G. Then $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup (resp. left ideal, right ideal, lateral ideal) of G if and only if $\tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])}$ is a ternary subsemigroup (resp. left ideal, right ideal, lateral ideal) of G, for all $[\lambda_1,\lambda_2],[\theta_1,\theta_2]\in[I]$.*

Proof: Assume that every non-empty level subset of $\tilde{A}$ is a ternary subsemigroup of G. Let $x,y,z\in G$ be such that

$$\bar{T}(\bar{M}_A(x),\bar{M}_A(y),\bar{M}_A(z)) > \bar{M}_A(xyz)$$
$$\text{and } \bar{S}(\bar{N}_A(x),\bar{N}_A(y),\bar{N}_A(z)) < \bar{N}_A(xyz).$$

Choose $[\lambda_1,\lambda_2],[\theta_1,\theta_2]\in[I]$ such that

$$\bar{T}(\bar{M}_A(x),\bar{M}_A(y),\bar{M}_A(z)) \geq [\lambda_1,\lambda_2] > \bar{M}_A(xyz)$$

and

$$\bar{S}(\bar{N}_A(x),\bar{N}_A(y),\bar{N}_A(z)) \leq [\theta_1,\theta_2] < \bar{N}_A(xyz)$$

This implies that $x,y,z\in\tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])}$ but $xyz\notin\tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])}$. Which is a contradiction. Hence

$$\bar{T}(\bar{M}_A(x),\bar{M}_A(y),\bar{M}_A(z)) \leq \bar{M}_A(xyz)$$
$$\text{and } \bar{S}(\bar{N}_A(x),\bar{N}_A(y),\bar{N}_A(z)) \geq \bar{N}_A(xyz).$$

Thus $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G.

Conversely, assume that $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G. Let $x,y,z \in \tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])}$. Then $\bar{M}_A(x) \geq [\lambda_1,\lambda_2]$, $\bar{M}_A(y) \geq [\lambda_1,\lambda_2]$, $\bar{M}_A(z) \geq [\lambda_1,\lambda_2]$ and $\bar{N}_A(x) \leq [\theta_1,\theta_2]$, $\bar{N}_A(y) \leq [\theta_1,\theta_2]$, $\bar{N}_A(z) \leq [\theta_1,\theta_2]$. Since

$$\bar{M}_A(xyz) \geq \bar{T}(\bar{M}_A(x),\bar{M}_A(y),\bar{M}_A(z))$$
$$\text{and } \bar{N}_A(xyz) \leq \bar{S}(\bar{N}_A(x),\bar{N}_A(y),\bar{N}_A(z)),$$

so,

$$\bar{M}_A(xyz) \geq \bar{T}([\lambda_1,\lambda_2],[\lambda_1,\lambda_2],[\lambda_1,\lambda_2]) = [\lambda_1,\lambda_2]$$
$$\text{and } \bar{N}_A(xyz) \leq \bar{S}([\theta_1,\theta_2],[\theta_1,\theta_2],[\theta_1,\theta_2]) = [\theta_1,\theta_2].$$

This implies that $\bar{M}_A(xyz) \geq [\lambda_1,\lambda_2]$ and $\bar{N}_A(xyz) \leq [\theta_1,\theta_2]$. Thus $xyz \in \tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])}$ and hence, $\tilde{A}_{([\lambda_1,\lambda_2],[\theta_1,\theta_2])}$ is a ternary subsemigroup of G. The other cases can seen in a similar way.

Theorem 3.15: *A non-empty subset B of a ternary semigroup G is a ternary subsemigroup of G if and only if the interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy subset $\tilde{A}=(\bar{M}_A,\bar{N}_A)$ defined by*

$$\bar{M}_A(x)=\begin{cases}[a_1,a_2] & \text{if } x\in G-B\\ [a_3,a_4] & \text{if } x\in B\end{cases}$$
$$\text{and } \bar{N}_A(x)=\begin{cases}[b_1,b_2] & \text{if } x\in G-B\\ [b_3,b_4] & \text{if } x\in B\end{cases}$$

is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G, where $[0,0]\leq[a_1,a_2]<[a_3,a_4]$ and $[0,0]\leq[b_3,b_4]<[b_1,b_2]$.

Proof: The proof is straightforward.

Theorem 3.16: *A non-empty subset B of a ternary semigroup G is a left (right, lateral) ideal of G if and only if the interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy subset $\tilde{A}=(\bar{M}_A,\bar{N}_A)$ defined by*

$$\bar{M}_A(x)=\begin{cases}[a_1,a_2] & \text{if } x\in G-B\\ [a_3,a_4] & \text{if } x\in B\end{cases}$$
$$\text{and } \bar{N}_A(x)=\begin{cases}[b_1,b_2] & \text{if } x\in G-B\\ [b_3,b_4] & \text{if } x\in B\end{cases}$$

is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideal of G, where $[0,0]\leq[a_1,a_2]<[a_3,a_4]$ and $[0,0]\leq[b_3,b_4]<[b_1,b_2]$.

Proof: Let B be a left ideal of G. Now, let $x,y,z\in G$. Let $z\in B$. Then $xyz\in B$. Hence

$$\bar{M}_A(xyz)=[a_3,a_4]=\bar{M}_A(z)$$
$$\text{and } \bar{N}_A(xyz)=[b_3,b_4]=\bar{N}_A(z).$$

If $z\notin B$, then

$$\bar{M}_A(z)=[a_1,a_2]\leq\bar{M}_A(xyz)$$
$$\text{and } \bar{N}_A(z)=[b_1,b_2]\geq\bar{N}_A(xyz).$$

Hence $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of G.

Conversely, assume that $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of G. If $z\in B$ and $x,y\in G$, then

$$\bar{M}_A(xyz)\geq\bar{M}_A(z)=[a_3,a_4]$$
$$\text{and } \bar{N}_A(xyz)\leq\bar{N}_A(z)=[b_3,b_4].$$

This implies that $\bar{M}_A(xyz)=[a_3,a_4]$ and $\bar{N}_A(xyz)=[b_3,b_4]$, that is $xyz\in B$. Hence B is a left ideal of G.

Definition 3.17: *Let G be a ternary semigroup and let $\phi\neq A\subseteq G$. Then interval-valued intuitionistic fuzzy characteristic function $\tilde{\chi}_A=(\bar{M}_{\chi_A},\bar{N}_{\chi_A})$ of A is defined as*

$$\bar{M}_{\chi_A}(x)=\begin{cases}[1,1] & \text{if } x\in A\\ [0,0] & \text{if } x\notin A\end{cases}$$
$$\text{and } \bar{N}_{\chi_A}(x)=\begin{cases}[0,0] & \text{if } x\in A\\ [1,1] & \text{if } x\notin A\end{cases}$$

Theorem 3.18: *Let A be a non-empty subset of a ternary semigroup G. Then A is a ternary subsemigroup of G if and only if $\tilde{\chi}_A$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G.*

Proof: Let A be a ternary subsemigroup of G. For any $x,y,z\in G$, we have the following cases:

Case (1): If $x,y,z\in A$, then $xyz\in A$. since A is a ternary subsemigroup of G. Then $\bar{M}_{\chi_A}(xyz)=[1,1]$, $\bar{M}_{\chi_A}(x)=[1,1]$, $\bar{M}_{\chi_A}(y)=[1,1]$ and $\bar{M}_{\chi_A}(z)=[1,1]$. Therefore

$$\bar{M}_{\chi_A}(xyz)=\bar{T}(\bar{M}_{\chi_A}(x),\bar{M}_{\chi_A}(y),\bar{M}_{\chi_A}(z)).$$

and $\bar{N}_{\chi_A}(xyz)=[0,0]$, $\bar{N}_{\chi_A}(x)=[0,0]$, $\bar{N}_{\chi_A}(y)=[0,0]$ and $\bar{N}_{\chi_A}(z)=[0,0]$. Therefore

$$\bar{N}_{\chi_A}(xyz)=\bar{S}(\bar{N}_{\chi_A}(x),\bar{N}_{\chi_A}(y),\bar{N}_{\chi_A}(z)).$$

Case (2): If $x \notin A$ or $y \notin A$ or $z \notin A$, then $\bar{M}_{\chi_A}(x) = [0,0]$ or $\bar{M}_{\chi_A}(y) = [0,0]$ or $\bar{M}_{\chi_A}(z) = [0,0]$. So

$$\bar{M}_{\chi_A}(xyz) \geq [0,0] = \bar{T}\left(\bar{M}_{\chi_A}(x), \bar{M}_{\chi_A}(y), \bar{M}_{\chi_A}(z)\right).$$

and $\bar{N}_{\chi_A}(x) = [1,1]$ or $\bar{N}_{\chi_A}(y) = [1,1]$ or $\bar{N}_{\chi_A}(z) = [1,1]$. So

$$\bar{N}_{\chi_A}(xyz) \leq [1,1] = \bar{S}\left(\bar{N}_{\chi_A}(x), \bar{N}_{\chi_A}(y), \bar{N}_{\chi_A}(z)\right).$$

Hence, $\tilde{\chi}_A$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G.

Conversely, suppose $\tilde{\chi}_A$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G and let $x, y, z \in A$. Then, we have

$$\bar{M}_{\chi_A}(xyz) \geq \bar{T}\left(\bar{M}_{\chi_A}(x), \bar{M}_{\chi_A}(y), \bar{M}_{\chi_A}(z)\right) = [1,1]$$

$$\bar{M}_{\chi_A}(xyz) \geq [1,1] \text{ but } \bar{M}_{\chi_A}(xyz) \leq [1,1]$$

Thus $\bar{M}_{\chi_A}(xyz) = [1,1]$.

and

$$\bar{N}_{\chi_A}(xyz) \leq \bar{S}\left(\bar{N}_{\chi_A}(x), \bar{N}_{\chi_A}(y), \bar{N}_{\chi_A}(z)\right) = [0,0]$$

$$\bar{N}_{\chi_A}(xyz) \leq [0,0] \text{ but } \bar{N}_{\chi_A}(xyz) \geq [0,0]$$

Thus $\bar{N}_{\chi_A}(xyz) = [0,0]$.

Hence $xyz \in A$. Therefore A is a ternary subsemigroup of G.

Theorem 3.19: *Let A be a non-empty subset of a ternary semigroup G. Then A is a left (right, lateral) ideal of G if and only if $\tilde{\chi}_A$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$ -fuzzy left (right, lateral) ideal of G.*

Proof: The proof is straightforward.

Definition 3.20: *A mapping $\eta : [I] \to [I]$ is called a negation if it satisfies*

(1) $\eta([0,0]) = [1,1]$, $\eta([1,1]) = [0,0]$,
(2) η is non-increasing,
(3) $\eta(\eta(\bar{x})) = \bar{x}$.

Remark 3.21: *The interval t-norm and interval s-conorm are said to be dual with respect to the negation $\eta(\bar{x}) = [1,1] - \bar{x}$, if*

$$\bar{T}(\bar{x}, \bar{y}) = \bar{T}\left(\eta\left(\eta((\bar{x}), \eta((\bar{y})\right)\right).$$

This holds with respect to η if and only if $\bar{S}(\bar{x}, \bar{y}) = \bar{S}\left(\eta\left(\eta((\bar{x}), \eta((\bar{y})\right)\right)$.

Theorem 3.22: *Let $\tilde{A} \in IIF(G,S,T)$. If $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$ -fuzzy ternary subsemigroup of G, then*

(1) $\Box\ \tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$ -fuzzy ternary subsemigroup of G.
(2) $\Diamond\ \tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$ -fuzzy ternary subsemigroup of G.

Proof: (1) It is sufficient to prove that for all $x, y, z \in G$, $\bar{M}_A^*$ satisfies

$$\bar{M}_A^*(xyz) \leq \bar{S}(\bar{M}_A^*(x), \bar{M}_A^*(y), \bar{M}_A^*(z)).$$

Now let $x, y, z \in G$, we have

$$\begin{aligned} \bar{M}_A^*(xyz) &= [1,1] - \bar{M}_A(xyz) \\ &\leq [1,1] - \left\{\bar{T}\left(\bar{M}_A(x), \bar{M}_A(y), \bar{M}_A(z)\right)\right\} \\ &= \eta\left\{\bar{T}\left(\bar{M}_A(x), \bar{M}_A(y), \bar{M}_A(z)\right)\right\} \\ &= \bar{S}\left(\eta\bar{M}_A(x), \eta\bar{M}_A(y), \eta\bar{M}_A(z)\right) \\ &= \bar{S}\left([1,1] - \bar{M}_A(x), [1,1] - \bar{M}_A(y), [1,1] - \bar{M}_A(z)\right) \\ &= \bar{S}(\bar{M}_A^*(x), \bar{M}_A^*(y), \bar{M}_A^*(z)). \end{aligned}$$

This shows that $\bar{M}_A^*(xyz) \leq \bar{S}(\bar{M}_A^*(x), \bar{M}_A^*(y), \bar{M}_A^*(z))$.

(2) It is sufficient to prove that for all $x, y, z \in G$, $\bar{N}_A^*$ satisfies

$$\bar{N}_A^*(xyz) \geq \bar{T}(\bar{N}_A^*(x), \bar{N}_A^*(y), \bar{N}_A^*(z)).$$

Now, let $x, y, z \in G$, we have

$$\begin{aligned} \bar{N}_A^*(xyz) &= [1,1] - \bar{N}_A(xyz) \\ &\geq [1,1] - \left\{\bar{S}\left(\bar{N}_A(x), \bar{N}_A(y), \bar{N}_A(z)\right)\right\} \\ &= \eta\left\{\bar{S}\left(\bar{N}_A(x), \bar{N}_A(y), \bar{N}_A(z)\right)\right\} \\ &= \bar{T}\left(\eta\bar{N}_A(x), \eta\bar{N}_A(y), \eta\bar{N}_A(z)\right) \\ &= \bar{T}\left([1,1] - \bar{N}_A(x), [1,1] - \bar{N}_A(y), [1,1] - \bar{N}_A(z)\right) \\ &= \bar{T}(\bar{N}_A^*(x), \bar{N}_A^*(y), \bar{N}_A^*(z)). \end{aligned}$$

This shows that $\bar{N}_A^*(xyz) \geq \bar{T}(\bar{N}_A^*(x), \bar{N}_A^*(y), \bar{N}_A^*(z))$.

Theorem 3.23: *Let $\tilde{A} \in IIF(G,S,T)$. If $\tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$ -fuzzy ternary subsemigroup of G, then*

(1) $\Box\ \tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G.
(2) $\Diamond\ \tilde{A}$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G.

Proof: The proof is straightforward.

Theorem 3.24: *Let $\tilde{A}, \tilde{B}, \tilde{C} \in IIF(G,S,T)$. Let G is regular then $\bar{M}_{A \circ B \circ C} = \bar{T}(\bar{M}_A, \bar{M}_B, \bar{M}_C)$ and $\bar{N}_{A \circ B \circ C} =$*

$\bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)$ *for every interval-valued intuitionistic* $(\bar{S},\bar{T})$*-fuzzy right ideal* $\tilde{A}$, *every interval-valued intuitionistic* $(\bar{S},\bar{T})$*-fuzzy lateral ideal* $\tilde{B}$ *and every interval-valued intuitionistic* $(\bar{S},\bar{T})$*-fuzzy left ideal* $\tilde{C}$ *of G.*

Proof: Let $\tilde{A}$ be an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy right ideal, $\tilde{B}$ an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy lateral ideal and $\tilde{C}$ an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left ideal of G. Then $\bar{M}_{A\circ B\circ C} \le \bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)$ and $\bar{N}_{A\circ B\circ C} \ge \bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)$. Let $a \in S$. Then there exists $x \in S$ such that $a = axa = a(xax)a$. Thus we have

$$\begin{aligned}\bar{M}_{A\circ B\circ C}(a) &= \underset{a=xyz}{\vee} \bar{T}\left(\bar{M}_A(x),\bar{M}_B(y),\bar{M}_C(z)\right)\\ &\ge \bar{T}\left(\bar{M}_A(a),\bar{M}_B(xax),\bar{M}_C(a)\right)\\ &\ge \bar{T}\left(\bar{M}_A(a),\bar{M}_B(a),\bar{M}_C(a)\right)\\ &= \bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)(a).\end{aligned}$$

and,

$$\begin{aligned}\bar{N}_{A\circ B\circ C}(a) &= \underset{a=xyz}{\wedge} \bar{S}\left(\bar{N}_A(x),\bar{N}_B(y),\bar{N}_C(z)\right)\\ &\le \bar{S}\left(\bar{N}_A(a),\bar{N}_B(xax),\bar{N}_C(a)\right)\\ &\le \bar{S}\left(\bar{N}_A(a),\bar{N}_B(a),\bar{N}_C(a)\right)\\ &= \bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)(a).\end{aligned}$$

So, we get $\bar{M}_{A\circ B\circ C} \ge \bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)$ and $\bar{N}_{A\circ B\circ C} \le \bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)$. Hence $\bar{M}_{A\circ B\circ C} = \bar{T}(\bar{M}_A,\bar{M}_B,\bar{M}_C)$ and $\bar{N}_{A\circ B\circ C} = \bar{S}(\bar{N}_A,\bar{N}_B,\bar{N}_C)$.

4. Homomorphic Images and Preimages

In this section, we discuss some properties of homomorphic image and preimage of interval valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideals of ternary semigroups.

Definition 4.1: *[50] Let X and Y be two given ordinary sets and $f: X \to Y$ be a function. Let $\tilde{A} \in IIF(X)$ and $\tilde{B} \in IIF(Y)$. Then the image and preimage is defined by $f(\tilde{A})(y) = (f(\bar{M}_A)(y), f(\bar{N}_A)(y))$ and $f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$, where $\forall x, y \in Y$*

$$\begin{aligned}f(\tilde{A})(y) &= \left[\underset{x\in f^{-1}(y)}{\vee}(\bar{M}_A)(x), \underset{x\in f^{-1}(y)}{\wedge}(\bar{N}_A)(x)\right]\\ &= \left[[f(M_A^-)(y), f(M_A^+)(y)],[f(N_A^-)(y), f(N_A^+)(y)]\right],\end{aligned}$$

$$\begin{aligned}f^{-1}(\tilde{B})(x) &= \left[[f^{-1}(M_B^-)(x), f^{-1}(M_B^+)(x)],[f^{-1}(N_B^-)(x), f^{-1}(N_B^+)(x)]\right]\\ &= \left[[M_B^-(f(x)), M_B^+(f(x))],[N_B^-(f(x)), N_B^+(f(x))]\right],\end{aligned}$$

Theorem 4.2: *Let $f: G_1 \to G_2$ be a homomorphism from a ternary semigroup G_1 to a ternary semigroup G_2. If $\tilde{A} = (\bar{M}_A,\bar{N}_A)$ is an an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G_2, then the preimage $f^{-1}(\tilde{A}) = \left(f^{-1}(\bar{M}_A), f^{-1}(\bar{N}_A)\right)$ of $\tilde{A}$ under f is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G_1.*

Proof: Let $\tilde{A} = (\bar{M}_A,\bar{N}_A)$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup of G_2 and let $x, y, z \in G_1$. Then we have

$$\begin{aligned}f^{-1}(\bar{M}_A(xyz)) &= \bar{M}_A\left(f(xyz)\right) = \bar{M}_A\left(f(x)f(y)f(z)\right)\\ &\ge \bar{T}(\bar{M}_A f(x), \bar{M}_A f(y), \bar{M}_A f(z))\\ &= \bar{T}\begin{pmatrix} f^{-1}(\bar{M}_A(x)),\\ f^{-1}(\bar{M}_A(y)), f^{-1}(\bar{M}_A(z))\end{pmatrix}.\end{aligned}$$

and,

$$\begin{aligned}f^{-1}(\bar{N}_A(xyz)) &= \bar{N}_A\left(f(xyz)\right) = \bar{N}_A\left(f(x)f(y)f(z)\right)\\ &\ge \bar{S}(\bar{N}_A f(x), \bar{N}_A f(y), \bar{N}_A f(z))\\ &= \bar{S}\begin{pmatrix} f^{-1}(\bar{N}_A(x)),\\ f^{-1}(\bar{N}_A(y)), f^{-1}(\bar{N}_A(z))\end{pmatrix}.\end{aligned}$$

This shows that $f^{-1}(\tilde{A}) = \left(f^{-1}(\bar{M}_A), f^{-1}(\bar{N}_A)\right)$ is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy ternary subsemigroup G_1.

Theorem 4.3: *Let $f: G_1 \to G_2$ be a homomorphism from a ternary semigroup G_1 to a ternary semigroup G_2. If $\tilde{A} = (\bar{M}_A,\bar{N}_A)$ is an an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideal of G_2, then the preimage $f^{-1}(\tilde{A}) = \left(f^{-1}(\bar{M}_A), f^{-1}(\bar{N}_A)\right)$ of $\tilde{A}$ under f is an interval-valued intuitionistic $(\bar{S},\bar{T})$-fuzzy left (right, lateral) ideal of G_1.*

Proof: The proof is straightforward.

5. Conclusion

We introduced the notion of the interval valued intuitionistic fuzzy ternary semigroup with respect to interval t-norm $\bar{T}$ and interval t-conorm $\bar{S}$ and we studied several properties. In addition, we provided relationship between interval valued intuitionistic fuzzy ternary semigroups and $([\lambda_1, \lambda_2],[\theta_1, \theta_2])$-level subsets.

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