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Explicit and Implicit Methods for Fractional Diffusion Equations with the Riesz Fractional Derivative

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Abstract

In this paper, a fractional diffusion equation by using the explicit numerical method in a finite domain with second-order accuracy which includes the Riesz fractional derivative approximation is studied. For the Riesz fractional derivative approximation, "fractional centered derivative" approach is used. The error of the Riesz fractional derivative to the fractional centered difference is calculated. We used the implicit numerical method to solve the fractional diffusion equation and also investigated the stability of explicit and implicit methods. The maximum error of the implicit method for fractional diffusion equation with using fractional centered difference approach is shown by using the numerical results.

Keywords: Riesz Fractional Derivative Operator, Implicit Method, Explicit Method, Fractional Central Difference.
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1. Introduction

Fractional differential equations are used frequently in science and engineering, such as: fractional diffusion and wave equations [1, 2], electrical systems [3], viscoelasticity theory [3], control systems [3], biomedical engineering [4] finance [5] and the economic analysis of the stock prices. Let the $\frac{\partial^\alpha}{\partial |x|^\alpha}$ be the Riesz fractional derivative operator for $1 < \alpha \le 2$ that is defined in [6–8] as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{-\infty}^{\infty}\left|x-\xi\right|^{1-\alpha} u(\xi,t)\,d\xi \tag{1}$$

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

We consider the following equation in a finite domain associated with initial and Dirichlet boundary conditions

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t), \quad a<x<b, \quad 0<t\le T \tag{2}$$

$$\begin{cases} u(a,t) = \varphi(t), & 0<t\le T \\ u(b,t) = \Phi(t), & 0<t\le T \\ u(x,0) = \psi(x), & a\le x\le b \end{cases} \tag{3}$$

where $D > 0$ is diffusion coefficient, and $f(x,t)$, $\varphi(t)$, $\Phi(t)$, $\psi(x)$ are sufficiently smooth functions. To estimate the Riesz fractional derivatives, Grunwald–Letnikov derivative approximation, we use scheme of order O(h) [9–15]. Meerschaert and Tadjeran [11] and Tadjeranetal. [12] applied the Crank–Nicolson method with Grunwald–Letnikov

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derivative approximation to a linear diffusion equation, which has an independent fractional derivative, represented that the used method is unconditionally stable for given problems. Shen and et al. [13] applied implicit and explicit finite difference methods with Grunwald–Letnikov derivative approximation to a linear Riesz fractional diffusion equation, and showed that the explicit method is conditionally and the implicit models is unconditionally stable for given problems. To improve the convergence order of used approximation, they used the Richordsons extrapolation [11–14]. Zhang and Liu [15] applied the implicit finite difference method with Grunwald–Letnikov derivative approximation to a nonlinear Riesz fractional diffusion equation and showed that the used method is stable for small time. As a new approach, Ortigueire [16] defined the "fractional central derivative" and proved that the Riesz fractional derivative of an analytic function can be represented by fractional central derivative.

In Section 2, we show that the fractional central difference is approximated with $O(h^2)$ accurate to the Riesz fractional derivative for $1 < \alpha \le 2$. In Section 3 and 5 we applied the explicit and implicit method for the problem (2) and (3) by using the fractional centered difference discretization. In Section 4 and 6 we give the stability properties of the explicit and implicit method for the problem (2) and (3). Finally, in the last section we presented a numerical solution of an example by using the implicit method.

2. Approximation by Fractional Centered Difference

In [16] for $\alpha > -1$ the fractional centered difference is defined by

$$\Delta_h^{\alpha}\theta(x)=\sum_{k=-\infty}^{\infty}\frac{(-1)^k\,\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)}\theta(x-kh) \quad (4)$$

and it is shown that

$$\lim_{h\to 0}\frac{\Delta_h^{\alpha}\theta(x)}{h^{\alpha}}=\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=-\infty}^{\infty}\frac{(-1)^k\,\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)}\theta(x-kh) \quad (5)$$

represents the Riesz fractional derivative (1) for the case of $1< \alpha \le 2$. we use Eq. (5) as a discretization to the Riesz fractional derivative. Therefore, we express the following property and lemma.

Property 2.1.
Let $g_k=\dfrac{(-1)^k\,\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)}$ be the coefficients of the centered finite difference approximation (5) for $k = 0, \pm1,\pm2,\ldots$, and $\alpha>-1$. Then

$$g_k\ge 0$$

$$g_k=g_{-k}\le 0 \text{ for } |k|\ge 1 \quad (6)$$

Lemma 2.2.
Let $f\in C^5(R)$ and all derivatives up to order five belong to $L_1(R)$ and

$$\Delta_h^{\alpha}f(x)=\sum_{k=-\infty}^{\infty}\frac{(-1)^k\,\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)}f(x-kh) \quad (7)$$

be the fractional centered difference. Then

$$-h^{-\alpha}\Delta_h^{\alpha}f(x)=\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}}+O\left(h^2\right) \quad (8)$$

When $h\to 0$ and $\dfrac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}}$ is the Riesz fractional derivative for $1<\alpha\le 2$.

if f^* is defined by

$$f^*(x)=\begin{cases} f(x), & x\in[a,b] \\ 0, & x\notin[a,b]\end{cases}$$

such that $f^*(x)\in C^5(\mathrm{R})$ and all derivatives up to order five belong to $L_1(R)$ then from Lemma 2.2, we have

$$\frac{\partial^{\alpha}f^*(x)}{\partial|x|^{\alpha}}=-h^{-\alpha}\sum_{k=-\infty}^{\infty}g_k f^*(x-kh)+O\left(h^2\right)$$

Since $f^*(x)=0$ for $x\notin[a,b]$ we get

$$\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}}=-h^{-\alpha}\sum_{k=\frac{b-x}{h}}^{\frac{x-a}{h}}g_k f(x-kh)+O\left(h^2\right) \quad (9)$$

Where $h=\dfrac{b-a}{m}$, and m is the number of partitions of the interval $[a, b]$.

3. The Explicit Discretization for Fractional Diffusion Equation

Explicit discretization for the equations 2 and 3 is

$$\frac{u_i^{j+1}-u_i^j}{\tau}=-Dh^{-\alpha}\sum_{k=-m+i}^{i}g_k u_{i-k}^j+f_i^{j+\frac{1}{2}} \quad (10)$$

Or

$$u_i^{j+1} = u_i^j - \frac{D_\tau}{h^\alpha}\sum_{k=-m+i}^{i} g_k u_{i-k}^j + \tau f_i^{j+\frac{1}{2}} \tag{11}$$

For $i = 1, 2, \ldots, m-1$, $j = 0, 1, \ldots, N-1$, $h = \frac{b-a}{m}, x_i = a + ih, \tau = \frac{T}{N}, t_j = j\tau$, and $u_i^j = u(x_i, t_j)$. We can rewrite the system (11) in matrix – vector form as

$$U^{j+1} = (I - A)U^j + BC + \tau F^{j+\frac{1}{2}} \tag{12}$$

where $U^j = (u_1^j, u_2^j, \ldots, u_{m-1}^j)^T$, $F^{j+\frac{1}{2}} = (f_1^{j+\frac{1}{2}}, f_2^{j+\frac{1}{2}}, \ldots, f_{m-1}^{j+\frac{1}{2}})^T$, $f_i^{j+\frac{1}{2}} = f(x_i, t_{j+\frac{1}{2}}), t_{j+\frac{1}{2}} = j\tau + \frac{\tau}{2}$ and A is an matrix (m–1) × (m–1) with the entries $a = \frac{\tau D}{(h^\alpha)}, g_k = g_{-k}$, and BC is the vector of boundary conditions.

$$A = \begin{pmatrix} ag_0 & ag_{-1} & ag_{-2} & \cdots & ag_{-m+2} \\ ag_1 & ag_0 & ag_{-1} & & ag_{-m+3} \\ ag_2 & ag_1 & ag_0 & & ag_{-m+4} \\ \vdots & \vdots & & \ddots & \\ ag_{m-2} & ag_{m-3} & ag_{m-4} & & ag_0 \end{pmatrix},$$

$$BC = a\begin{pmatrix} g_1u_0 + g_{-m+1}u_m \\ g_2u_0 + g_{-m+2}u_m \\ g_3u_0 + g_{-m+3}u_m \\ \vdots \\ g_{m-1}u_0 + g_{-1}u_m \end{pmatrix} \tag{13}$$

4. Stability of the Explicit Difference Approximation Method

Theorem 4.1 equation (12) for problem (2) and (3) is conditionally stable.

Proof. In [16] it was shown that the coefficients of the generating function (5) for z ∈ R has the following form:

$$|2\sin(z/2)|^\alpha = \sum_{k=-\infty}^{\infty} \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)} e^{ikh} \tag{14}$$

Let λ be the eigenvalue of matrix A, then by using (14) and Gerschgorin's circle theorem [19] we have:

$$|\lambda - ag_0| \le r_i = a\sum_{k=-m+i}^{i-1} |g_k| < ag_0 \tag{15}$$

Where $\sum_{k=-\infty}^{\infty} |g_k| = g_0$ that is, we have

$$0 < \lambda < 2ag_0 \tag{16}$$

For stability, the eigenvalues of matrix (I-A) should be satisfy | 1–λ | <1, therefore, 0 < λ < 2, Consequently $2ag_0 < 2$, $a = \frac{\tau}{h^\alpha} < \frac{1}{g_0}$ and discrete equation (12) is conditionally stable.

5. The Implicit Discretization for Fractional Diffusion Equation

Stability of explicit methods was studied in the previous section. Now we want to discuss the implicit method. Implicit discretization for the equations 2 and 3 is

$$\frac{u_i^{j+1} - u_i^j}{\tau} = -Dh^{-\alpha}\sum_{k=-m+i}^{i} g_k u_{i-k}^{j+1} + f_i^{j+\frac{1}{2}} \tag{17}$$

Or

$$u_i^{j+1} + \frac{D\tau}{h^\alpha}\sum_{k=-m+i}^{i} g_k u_{i-k}^{j+1} = u_i^j + \tau f_i^{j+\frac{1}{2}} \tag{18}$$

For $i = 1, 2, \ldots, m-1$, $j = 0, 1, \ldots, N-1$, $h = \frac{b-a}{m}, x_i = a + ih, \tau = \frac{T}{N}, t_j = j\tau$, and $u_i^j = u(x_i, t_j)$. We can rewrite the system (18) in matrix – vector form as

$$(I + A)U^{j+1} = U^j + BC + \tau F^{j+\frac{1}{2}} \tag{19}$$

where $U^j = (u_1^j, u_2^j, \ldots, u_{m-1}^j)^T, F^{j+\frac{1}{2}} = (f_1^{j+\frac{1}{2}}, f_2^{j+\frac{1}{2}}, \ldots, f_{m-1}^{j+\frac{1}{2}})^T$, $f_i^{j+\frac{1}{2}} = f(x_i, t_{j+\frac{1}{2}}), t_{j+\frac{1}{2}} = j\tau + \frac{\tau}{2}$, and A is an matrix (m–1) × (m–1) with the entries $a = \frac{\tau D}{h^\alpha}, g_k = g_{-k}$, and BC is the vector of boundary conditions.

$$A = \begin{pmatrix} ag_0 & ag_{-1} & ag_{-2} & \cdots & ag_{-m+2} \\ ag_1 & ag_0 & ag_{-1} & & ag_{-m+3} \\ ag_2 & ag_1 & ag_0 & & ag_{-m+4} \\ \vdots & \vdots & & \ddots & \\ ag_{m-2} & ag_{m-3} & ag_{m-4} & & ag_0 \end{pmatrix},$$

$$\tag{20}$$

$$BC = a\begin{pmatrix} (g_1u_0^j + g_{-m+1}u_m^j) - (g_1u_0^{j+1} + g_{-m+1}u_m^{j+1}) \\ (g_2u_0^j + g_{-m+2}u_m^j) - (g_2u_0^{j+1} + g_{-m+2}u_m^{j+1}) \\ (g_3u_0^j + g_{-m+3}u_m^j) - (g_3u_0^{j+1} + g_{-m+3}u_m^{j+1}) \\ \vdots \\ (g_{m-1}u_0^j + g_{-1}u_m^j) - (g_{m-1}u_0^{j+1} + g_{-1}u_m^{j+1}) \end{pmatrix}$$

6. Stability of the Explicit Difference Approximation Method

Theorem 6.1 discrete equation (18) for problems (2) and (3) is unconditionally stable.

Proof: Let λ is the eigenvalue of matrix A, then by using (14) and Gerschgorin's circle theorem [16] we have:

$$\left|\lambda - ag_0\right| \le r_i = a \sum_{k=-m+i}^{i-1} \left|g_k\right| < ag_0 \tag{21}$$

where $\sum_{k=-\infty}^{\infty} \left|g_k\right| = g_0$ that is, we have

$$0 < \lambda < 2ag_0 \tag{22}$$

Hence the eigenvalues of the matrix $(I+A)^{-1}$ satisfy $\frac{1}{|1+\lambda|} < 1$ therefore, the spectral radius is less than one. Thus, the discrete equation (22) is unconditionally stable.

7. Numerical Example

For example, consider

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t), \quad a < x < b, \quad 0 < t \le T \tag{23}$$

With the initial and boundary conditions

$$\begin{cases} u(x,0) = x^2(1-x)^2, & 0 < x \le 1 \\ u(0,t) = 0, & 0 < t \le T \\ u(1,t) = 0, & 0 < t \le T \end{cases} \tag{24}$$

and with the non-homogeneous part

$$f(x,t) = (1+t)^{-1+\alpha}(-1+x)^2 x^2 \alpha + \frac{1}{\Gamma(5-\alpha)} x^{-\alpha} \left(\begin{array}{l} \left(\frac{1+t}{1-x}\right)^{\alpha} (-1+x)^2 x^{\alpha} \\ (12x^2 - 6x\alpha + (-1+\alpha)\alpha) + \\ (1+t)^{\alpha} x^2 \left(\begin{array}{l} 12(-1+x)^2 + \\ (-7+6x)\alpha + \alpha^2 \end{array} \right) \end{array} \right) \sec\left(\frac{\pi\alpha}{2}\right) \tag{25}$$

where the exact solution is

$$u(x,t) = (t+1)^{\alpha} x^2 (1-x)^2 \tag{26}$$

Figure 1 shows the solution profiles for the approximate solution computed using $a = 1.2, 1.4, 1.8$ at $t = 1$.

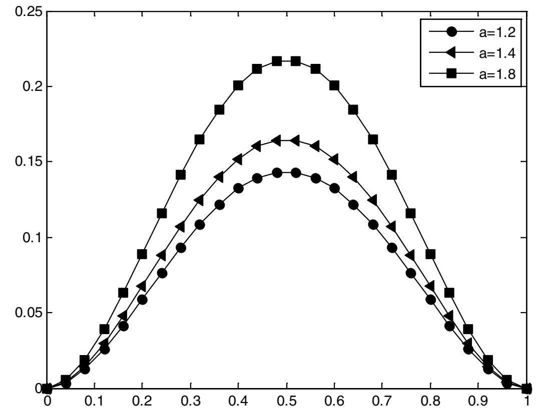

Figure 1. The exact solution of the Eqs. (23) with the initial and boundary conditions (24) at $t = 1$, $a = 1.2, 1.4, 1.8$.

Table 1. The maximum of errors for different values of α for $\tau = 0.01$, $t = 1$

α	h=0.01	h=0.005
1.2	3.8857×10^{-4}	4.0142×10^{-4}
1.4	5.2263×10^{-4}	5.4080×10^{-4}
1.6	6.8416×10^{-4}	7.1023×10^{-4}
1.8	8.7656×10^{-4}	9.1698×10^{-4}

Table 2. The maximum of errors for different values of α for $\tau = 0.01$, $t = 5$

α	h=0.01	h=0.005
1.2	4.7954×10^{-4}	5.3553×10^{-4}
1.4	7.8244×10^{-4}	8.7835×10^{-4}
1.6	1.2×10^{-3}	1.4×10^{-3}
1.8	1.9×10^{-3}	2.2×10^{-3}

In Table 1, we calculated the maximum errors of the implicit method for several different values of α in interval $1 < \alpha \le 2$ at $t = 1$ and $\tau = 0.01$ when $h = 0.01$ and $h = 0.005$.

In Table 2, we calculated the maximum errors of the implicit method for several different values of α in interval $1 < \alpha \le 2$ at $t = 5$ and $\tau = 0.01$, when $h = 0.01$ and $h = 0.005$. Comparing the two tables shows that accuracy of this method in $t = 1$ is more than in the time $t = 5$. When τ is considered as a constant, the accuracy of this method decrease in $\frac{h}{2}$.

In Table 3, we consider h as a constant and obtain the maximum error of method for $\tau = 0.01$ and $\tau = 0.005$ in $x = 0.4$.

Table 3. The maximum of errors for different values of α for h =0.01, x =0.4

α	$\tau = 0.01$	$\tau = 0.005$
1.2	3.5724×10^{-4}	1.7320×10^{-4}
1.4	4.8062×10^{-4}	2.3164×10^{-4}
1.6	6.2895×10^{-4}	3.0020×10^{-4}
1.8	8.7656×10^{-4}	3.7824×10^{-4}

Table 4. The maximum of errors for different values of α for h = 0.01, x = 0.7

α	τ =0.01	τ =0.005
1.2	2.7136×10^{-4}	1.3202×10^{-4}
1.4	3.6542×10^{-4}	1.7638×10^{-4}
1.6	4.7753×10^{-4}	2.2754×10^{-4}
1.8	6.0919×10^{-4}	2.8423×10^{-4}

In Table 4 we consider h as a constant and obtain the maximum error of method for τ = 0.01 and τ = 0.005 in x = 0.7. Comparing the two Tables 3, 4 shows that accuracy of this method is increases by halving the τ and accuracy of this method increase when x increase.

8. Conclusions

In this paper, Riesz derivative is approximated by using fractional central difference. Diffusion equation is approximated by Riesz derivative with applying the explicit and implicit numerical methods and we observed that after discretization for stability the spectral radius is less than one. The explicit method is conditionally stable and the stability of the implicit method is unconditional.

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