Some new types of stabilizers in BL-algebras and their applications

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Abstract

In this paper we introduce some new types of stabilizers in BL-algebras and we state and prove some theorems which determine the relations among stabilizers, MV-algebras and G"odel algebras. We define the concept of ZRS-condition in BL-algebras and we find a relation between this class of BL-algebras and MV-algebras. Finally, we show that the (semi) normal filters and fantastic filters are equal in BL-algebra.

Keywords: BL-algebra, MV-algebra, G"odel algebra, fantastic filter, stabilizer filter, normal filter.

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Introduction

BL-algebras are the algebraic structures for Hájek’s Basic Logic [7], in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, G"odel Logic and Product Logic. This Basic Logic (BL) is proposed as “the most general” many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0,1]. Most familiar example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. In 1958, Chang introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. MV-algebra, G"odel algebra and product algebra are the most known classes of BL-algebras. Hájek in [7] introduced the notions of filters and prime filters in BL-algebra and by using the prime filters of BL-algebras, he proved the completeness of BL. Filter theory plays an important role in studying these algebras. From logical point of view, various filter correspond to various set of provable formulas. Turunen [12, 13] studied some properties of filters and prime filters of BL-algebra (He called them deductive system and prime deductive system respectively). Now, in this paper we follow [3, 9], and we introduce some new types of stabilizers in BL-algebras and we state and prove some theorems which determine the relationship between stabilizers, MV-algebras and G"odel algebras. We prove that \( F(L) \) (the set of all filters on BL-algebras \( L \)) is a pseudocomplemented lattice. In the follow, we define the concept of ZRS-condition in BL-algebras and we show that, this class of BL-algebras are exactly MV-algebras and by using this condition we prove that the (semi) normal filters are exactly fantastic filters. Finally we answer to open problems that have been appeared in [3].

Preliminaries

Definition 2.1: [7] A BL-algebra is an algebra \((L, \wedge, \vee, \circ, \to, 0,1)\) with four binary operations \(\land, \lor, \circ, \to\) and two constant 0,1 such that

\begin{align*}
& (BL1) (L, \land, \lor, 0,1) \text{ is a bounded lattice,} \\
& (BL2) (L, \circ, 1) \text{ is a commutative monoid,} \\
& (BL3) c \leq a \to b \text{ if and only if } a \circ c \leq b, \text{ for all } a, b, c \in L, \\
& (BL4) a \land b = a \circ (a \to b), \\
& (BL5) (a \to b) \circ (b \to a) = 1.
\end{align*}

A BL-algebra is called a G"odel algebra, if \(a^2 = a \circ a = a\), for all \(a \in L\) and BL-algebra \(L\) is called an MV-algebra, if \((a^* )^* = a\) or equivalently \((a \to b) \to b = (b \to a) \to a\), for all \(a, b \in L\), where \(a^* = a \to 0\).

Lemma 2.2: [4, 5, 7] In each BL-algebra \(L\), the following relations hold for all \(a, b, c \in L\):

\begin{align*}
& (BL6) a \circ b \leq a, b, \quad a \circ b \leq a \land b, \quad a \circ 0 = 0, \\
& (BL7) a \leq b \text{ implies } c \leq b \circ c, \\
& (BL8) a \leq b \text{ if and only if } a \to b = 1, \\
& (BL9) 1 \to a = a, \quad a \to a = 1, \quad a \leq b \to a, \quad a \to 1 = 1, \\
& (BL10) a \circ a^* = 0, \quad 1 \circ a = a, \quad 0 \circ a = 1, \\
& (BL11) a \circ b = 0 \text{ if and only if } a \leq b^*, \\
& (BL12) a \to (b \circ c) = (a \circ b) \to c = b \to (a \circ c), \\
& (BL13) a \leq b \text{ then } b \to c \leq a \circ c \text{ and } c \leq a \circ b, \\
& (BL14) a \lor b = ((a \to b) \to b)) \land ((b \to a) \to a), \\
& (BL15) (a^* \circ a^* = 0, (a^* \circ a^*) = 1.
\end{align*}

Definition 2.3: [2, 5, 7, 8, 12] Let \(F\) be a non-empty subset of BL-algebra \(L\). Then:

(i) \(F\) is called an anti-filter of \(L\), if \(x, y \in F\) implies \(x \circ y \in F\) and \(x \in F, y \leq x \) imply \(y \in F\), for all \(x, y \in L\).

(ii) \(D\) is called a deductive system of \(L\), if \(1 \in D\) and if \(x \in D\) and \(x \to y \in D\), then \(y \in D\), for all \(x, y \in L\).

(iii) \(F\) is called a fantastic filter, if \(x \to F\) and \(z \to (x \to z) \in F\) and \(x \in F\) imply \((x \to y) \to y \to x \in F\), for all \(x, z \in L\).

(iv) \(F\) is called an anti-prime filter, if \((x \to F) \circ y \in F\) for all \(x, y \in L\).

(v) A filter \(F\) is called an anomalous filter, if \((y \to x) \to x \in F\), then \((x \to y) \to y \in F\), for all \(x, y \in L\).
Definition 2.4. [6] Let $L$ be a BL-algebra and $X \subseteq L$. The filter of $L$ generated by $X$ will be denoted by $< X >$. We have that $\leq \emptyset \implies 1$ and if $X \neq \emptyset$, $< X > = \{ y \in L | x_1 \cup \ldots \cup x_n \leq y \text{ for some } n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X \}$.

Theorem 2.5. [8, 11] Let $F$ be a filter of BL-algebra $L$. Then
(i) every filter of $L$ is a fantastic filter if and only if $L$ is an MV-algebra.
(ii) $F$ is a fantastic filter if and only if $\langle (x \to 0) \to 0 \rangle \to x \in F$, for all $x \in L$.
(iii) Let $F \subseteq G$ where $F$ be a fantastic filter and $G$ be a filter. Then $G$ is a fantastic filter.

Theorem 2.6. [7] Let $F$ be a filter of BL-algebra $L$. Then the binary relation $\equiv_F$ on $L$ which is defined by $x \equiv_F y$ if and only if $x \to y \in F$ and $y \to x \in F$ is a congruence relation on $L$. Define $\to, \cup, \cap, \setminus, \neg$ as all congruence classes of $\equiv_F$, as follows:
\[
[x \to y] = \{ x \to y \}, \quad [x \cup y] = \{ x \cup y \}, \quad [x \cap y] = \{ x \cap y \}
\]

Then $L^F = \langle (x \to 0) \to 0 \rangle \to x \in F$, for all $x \in L$. Then $G$, is a fantastic filter.

Theorem 2.7. [4, 14] Let $P$ be a proper filter of BL-algebra $L$. Then $P$ is a prime filter if and only if $F^P$ is a BL-chain.

Definition 2.8. [1] An element $a$ of lattice $L$ with bottom element $0$ is called a atom, if $0 \leq a$, where $x \leq y$ means that $x \prec y$ and there is no $z \in L \setminus \{ 0 \}$ such that $x \prec z \prec y$. An element $b$ of lattice $L$ with top element $1$ is called an coatom, if $b \preceq 1$.

Definition 2.9. [1] For lattice $L$ whith $0$ and $a \in L$, $b \in L$ is called the pseudocomplemented of $a$, if $a \wedge b = 0$ and for each $c \in L$, $c \wedge a = 0$ implies that $c \leq b$. A lattice in which every element has a pseudocomplemented is called a pseudocomplemented lattice.

Theorem 2.10. [11] Let $F$ be a filter of a BL-algebra $L$. Then we have $F$ is a fantastic filter if and only if the quotient algebra $L^F$ is an MV-algebra.

Note. From now on, in this paper we let $(L, \wedge, \vee, \circ, \to, 0, 1)$ or simply $L$ is a BL-algebra, unless otherwise stated.

Stabilizer filters in BL-algebras
In [9], the notion of stabilizer $X$, for any $\neq X \subseteq L$, have been defined by $\{ a \in L | a \to x = x, \forall x \in X \}$ and denoted by $\overline{X}$. But, in this paper we denoted $\overline{X}$ by $\overline{St_r(X)}$ and $\overline{St_\circ(X)}$ and we define the other notions such as $\overline{St_r(X)}$ and $\overline{St_\circ(X)}$.

Definition 3.1. Let $\neq X \subseteq L$. Then $\overline{St_r(X)}$ is defined as follows:
\[
\overline{St_r(X)} = \{ a \in L | x \to a = x, \forall x \in X \}
\]
Proof. (i) Let \( \overrightarrow{S}_I(X) = \{1\} \). Then by (BL9), \( x = 1 \rightarrow x = 1 \), for all \( x \in X \) and so \( X = \{1\} \). Now, let \( 1 \in X \). Then by (BL9), for any \( a \in \overrightarrow{S}_I(X) \), \( a = 1 \rightarrow a = 1 \) and so \( \overrightarrow{S}_I(X) = \{1\} \).

(ii) By (BL6), \( \overrightarrow{S}_r(\{1\}) = \{a \in L \mid 1 \rightarrow a = 1\} \) and by (BL6), \( \overrightarrow{S}_o(\{0\}) = \{a \in L \mid a \cap 0 = 0\} \).

(iii) Let \( \overrightarrow{S}_r(X) = L \). Since \( 0 \in L \), then \( 0 \in \overrightarrow{S}_o(X) \) and so for all \( x \in X \), \( x \cap 0 = 0 \). Now, by (BL6), \( x \cap 0 = 0 \) if \( 0 \cap x = 0 \). Hence, \( x = \{0\} \).

(iv) Let \( X \subseteq Y \) and \( a \in \overrightarrow{S}_o(Y) \). Then \( a \cap Y = y \), for all \( y \in Y \).

(v) Since \( X \cap Y \subseteq X \), then by (vi), \( \overrightarrow{S}_r(X) \subseteq \overrightarrow{S}_r(Y) \) and \( \overrightarrow{S}_o(Y) \subseteq \overrightarrow{S}_o(X) \).

Theorem 3.7. (i) If \( a \in L \) is an atom of \( L \), then \( \overrightarrow{S}_o(\{a\}) \) is a prime filter.

(ii) If \( b \in L \) is a co-atom of \( L \), then \( \overrightarrow{S}_o(\{b\}) \) is a prime filter.

Proof. (i) Let \( a \in L \) be an atom of \( L \). Hence, \( a \neq 0 \). We claim that \( 0 \notin \overrightarrow{S}_o(\{a\}) \).

Note, by (BL13), \( 0 \in \overrightarrow{S}_o(\{a\}) \).

(iii) Since \( 0 \notin \overrightarrow{S}_o(\{a\}) \), then by (BL6), \( \overrightarrow{S}_o(\{a\}) = \{a \in L \mid a \cap 0 = 0\} \).

Proof. (ii) Let \( b \in L \) be a co-atom of \( L \). Hence, \( b \neq 1 \). We claim that \( 0 \notin \overrightarrow{S}_o(\{b\}) \).

Note, by (BL13), \( 0 \in \overrightarrow{S}_o(\{b\}) \).

Note. \( \overrightarrow{S}_o(\{1\}) \) is not a filter of \( L \) in general.

Example 3.6. Let \( X = \{b\} \) in the Example 3.2(i). Then \( \overrightarrow{S}_r(X) = \{a\} \), which is not a filter of \( L \).

**Theorem 3.4.** \( L \) is a \( G \)-order algebra if and only if \( \{x\} \subseteq \overrightarrow{S}_o(\{x\}) \), for all \( x \in L \).

Proof: (i) \( L \) is a \( G \)-order algebra if and only if \( x \cap 0 = x \), for all \( x \in L \), and if and only if \( x \in \overrightarrow{S}_o(\{x\}) \), for all \( x \in L \).

**Theorem 3.5.** Let \( \overrightarrow{S}_r(X) \) and \( \overrightarrow{S}_r(X) \) be filters of \( L \).

Proof. Let \( b \in \overrightarrow{S}_r(X) \). Then \( a \rightarrow x = x \) and \( b \rightarrow x = x \), for all \( x \in X \).

Then \( \overrightarrow{S}_o(b) \rightarrow x \) for all \( b \in \overrightarrow{S}_o(X) \).

Note: \( \overrightarrow{S}_r(L) \) is not a filter of \( L \) in general.
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$\overline{St}_r (F)$. Therefore, $\overline{St}_r (F)$ is a pseudocomplemented of $F$.

**Corollary 3.11.** $(F(L), \wedge, \vee, \{1\}, L)$ is a pseudocomplemented lattice.

*Proof.* By Theorem 3.10, the proof is clear.

**BL-algebras with ZRS-condition**

**Definition 4.1.** We say BL-algebra $L$ satisfy Zero Right Stabilizer condition or briefly ZRS-condition if $\overline{St}_r (\{0\}) = \{1\}$.

**Example 4.2.**[10] Let $L = \{0,a,b,c,d,1\}$. Then $L$ by the following diagram is a bounded lattice.

![Diagram of a bounded lattice]

Now, let operations "→", "∗" and "⊙" on $L$ are defined as follows:

<table>
<thead>
<tr>
<th>→</th>
<th>0</th>
<th>a</th>
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<th>c</th>
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<tr>
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$x⊙y = \min\{x|x\leq y \rightarrow z\} = (x \rightarrow y)^*$

Then $(L, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a BL-algebra. Since $\overline{St}_r (\{0\}) = \{1\}$, then $L$ is a BL-algebra with ZRS-condition.

**Example 4.3.** BL-algebra the Example 3.2 (ii), does not satisfy in the ZRS-condition. Since $\overline{St}_r (\{0\}) = \{1,a,b,c\}$.

**Theorem 4.4.** L is an MV-algebra if and only if $L$ satisfies in the ZRS-condition.

*Proof.* $(\Rightarrow)$ Let $L$ be an MV-algebra and $x \in \overline{St}_r (\{0\})$. Then $x \rightarrow 0 = 0$ and so $x = (x^*)^* = (x \rightarrow 0) \rightarrow 0 = 0 \rightarrow 0 = 1$. Hence, $\overline{St}_r (\{0\}) = \{1\}$.

$(\Leftarrow)$ Let $L$ satisfies in the ZRS-condition. Then $\overline{St}_r (\{0\}) = \{1\}$. Now, let $x \in L$. Then by (BL12) $x \rightarrow x^* = x \rightarrow ((x \rightarrow 0) \rightarrow 0) = (x \rightarrow 0) \rightarrow (x \rightarrow 0) = 0$. Since by (BL15), $(x^* \rightarrow x)^* = 0$, then $(x^* \rightarrow x) \rightarrow 0 = 0$.

Hence, $x^* \rightarrow x \in \overline{St}_r (\{0\}) = \{1\}$. Therefore, $x^* \rightarrow x = 1$ and so $x^* = x$. Thus, $L$ is an MV-algebra.

**Proposition 4.5.** $L$ satisfies the ZRS-condition if and only if for any $x, y \in L$, $x \rightarrow y$ and $y \rightarrow x \in \overline{St}_r (\{0\})$, imply $x = y$.

*Proof.* $(\Rightarrow)$ Let $L$ satisfies in the ZRS-condition, $x, y \in L$ and $x \rightarrow y$ and $y \rightarrow x \in \overline{St}_r (\{0\})$. Since $\overline{St}_r (\{0\}) = \{1\}$, then $x \rightarrow y = y \rightarrow x = 1$, and so $x = y$.

$(\Leftarrow)$ Let $\in \overline{St}_r (\{0\})$. Since by (BL9), $x \rightarrow 1 = 1 \in \overline{St}_r (\{0\})$ and $1 \rightarrow x = x \in \overline{St}_r (\{0\})$, then by hypothesis, $x = 1$. Hence, $\overline{St}_r (\{0\}) = \{1\}$.

**Corollary 4.6.** (i) $L$ satisfies in the ZRS-condition if and only if every filter of $L$ is a fantastic filter.

(ii) $F$ is a fantastic filter if and only if $\overline{F}_r$ satisfies in the ZRS-condition.

(iii) If $L$ satisfies in the ZRS-condition, then for any filter of $L$, $\overline{F}_r$ satisfies in the ZRS-condition.

*Proof.* (i) $(\Rightarrow)$ Let $L$ satisfies in the ZRS-condition. Then by Theorem 4.4, $L$ is an MV-algebra and so by Theorem 2.5 (i), every filter of $L$ is a fantastic filter.

$(\Leftarrow)$ Let every filter of $L$ is a fantastic filter. Then by Theorem 2.5 (i), $L$ is an MV-algebra and so by Theorem 4.4, $L$ satisfies in the ZRS-condition.

(ii) By Theorem 2.10, $F$ be a fantastic filter of $L$ if and only if $\overline{F}_r$ is an MV-algebra. Also, by Theorem 4.4, $\overline{F}_r$ satisfies in the ZRS-condition if and only if $\overline{F}_r$ is an MV-algebra.

Therefore, $F$ is a fantastic filter if and only if $\overline{F}_r$ satisfies in the ZRS-condition.

(iii) Let $L$ satisfies in the ZRS-condition and $F$ be a filter of $L$. Let $x \in L$ such that $[x] \in \overline{St}_r (\{0\})$. Then $[x] \rightarrow [0] = [0]$ and so $[x \rightarrow 0] = [0]$ and this means that $(x^*)^* = (x \rightarrow 0) \rightarrow 0 \in F$. Since by Theorem 4.4, $L$ is an MV-algebra, then $(x^*)^* = x$ and so $x \in F$. Now, since $1 \in F$, then $1 \rightarrow x \in F$ and $x \rightarrow 1 \in F$ and this means that $[x] = [1]$. Hence, $\overline{St}_r (\{0\}) = \{1\}$. Therefore, $\overline{F}_r$ satisfies in the ZRS-condition.
Note: The converse of Corollary 4.6(iii) is not true in general. Consider following example:

**Example 4.7.** Let $L = \{0, a, b, 1\}$ be a chain such that $0 < a < b < 1$ and operation $\lor$ and $\rightarrow$ on $L$ are defined as follows:

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Then $(L, \land, \lor, \rightarrow, 0, 1)$ is a BL-algebra and it is clear that $F = \{b, 1\}$ is a fantastic filter. Now by Theorem 2.10, $L$ is a MV-algebra (satisfies in the ZRS-condition), but $L$ is not a MV-algebra. Since $(b^*)^* = 1$. Therefore, $L$ does not satisfy in the ZRS-condition.

(Semi) Normal filters and fantastic filters

In this section, we study a new class of filters that called semi normal filters. This is important for us, because this class of filters give a connection between normal filters and fantastic filter. In fact by the above definition, we solved the following open problems in [2].

**Open problem 1.** [2] Under what suitable conditions, a normal filter becomes a fantastic filter?

**Open problem 2.** [2] Under what suitable conditions, extension property for normal filter holds?

**Definition 5.1.** Let $F$ be a nontrivial filter of $L$. Then $F$ is called a semi-normal filter of $L$, if for all $x \in L$, $x \in F$ if and only if $\overline{x} \in F$.

**Example 5.2.** (i) Let $F = \{1, a, c\}$ in the Example 4.2. Then $F$ is a semi-normal filter. Since $a^* = a$, $c^* = c$ and $1^* = 1$.

(ii) In any MV-algebra, every filter is a semi-normal filter.

**Theorem 5.3.** Let $F$ be a normal (fantastic) filter of $L$. Then $F$ is a semi-normal filter of $L$.

Proof. Let $F$ be a normal filter and $x \in F$. Since $(x \rightarrow 0) \rightarrow (x \rightarrow 0) = 1$, then by (BL12), $x \rightarrow (x \rightarrow 0) \rightarrow 0 = 1$ and this means that $x \leq x^*$. Since $F$ is a filter and $x \in F$, then $x^* \in F$. Now, let $x^* \in F$. Hence, $(x \rightarrow 0) \rightarrow 0 \in F$ and since $F$ is normal filter then $(0 \rightarrow x) \rightarrow x \in F$. But, by (BL10), $(0 \rightarrow x) \rightarrow x = x$, hence $x \in F$. Therefore, $F$ is a semi-normal filter of $L$. Now, let $F$ be a fantastic filter of $L$ and $x \in F$. Similar to the proof of above, $x^* \in F$. Now, let $x \in L$ such that $x^* \in F$. By Theorem 2.5(ii), $x^* \rightarrow x \in F$. Since $x^* \in F$ and $F$ is a semi-normal filter of $L$, then $x \in F$. Hence, $F$ is a semi-normal filter of $L$.

**Theorem 5.4.** Let $F$ be a semi-normal filter of $L$. Then $F$ is an MV-algebra.

Proof. Let $F$ be a semi-normal filter of $L$. Since $F$ is a filter of $L$, then $F$ is a BL-algebra. We will show that $F$ satisfies in the ZRS-condition. Let $x \in \overline{F}$. Then $x \rightarrow 0 = 0$ and this means that $x^* = (x \rightarrow 0) \rightarrow 0 \in F$ and $F$ is a semi-normal filter of $L$, then $x \in L$. Hence, $x \rightarrow 1 = 1 \in F$ and $1 \rightarrow x \in F$ and so $x \in \overline{F}$. Thus, $\overline{F}$, $(\overline{\{0\}}) = \{\{1\}\}$ and so $F$ satisfies in the ZRS-condition. Therefore, by Theorem 4.4, $F$ is a MV-algebra.

Note: Now, in the following Theorem we answer to the open problems that been in [2].

**Theorem 5.5.** Let $F$ be a filter of $L$. Then $F$ is a normal filter if and only if $F$ is a fantastic filter.

Proof. $(\Rightarrow)$ Let $F$ be a normal filter. Then by Theorem 5.3, $F$ is a semi-normal filter and so by Theorem 5.4, $F$ is a MV-algebra. Now, by Theorem 2.10, $F$ is a fantastic filter.

Proof. $(\Leftarrow)$ Let $F$ be a fantastic filter of $L$. Then by Theorem 2.10, $F$ is an MV-algebra. But in MV-algebras any filter is a normal filter and since $(\overline{\{1\}})$ is a filter of $F$, then $(\overline{\{1\}})$ is a normal filter of $F$. Now, we show that $F$ is a normal filter of $L$. Let $(x \rightarrow y) \rightarrow y \in F$, for $x, y \in L$. Then $((x \rightarrow [y]) \rightarrow [y]) = [1]$ and so $((x \rightarrow [y]) \rightarrow [y]) \in \overline{\{1\}}$. Since $\overline{\{1\}}$ is a normal filter, then $((y \rightarrow [x]) \rightarrow [x]) = [1]$. Hence, $(y \rightarrow x) \rightarrow x \in F$ and this means that $F$ is a normal filter.

**Corollary 5.6.** (Extension property) Let $F$ and $G$ be filter of $L$, $F \subseteq G$ and $F$ be a normal filter. Then $G$ is a normal filter.

Proof. By Theorems 2.5(iii) and 5.5, the proof is clear.

**Corollary 5.7.** Let $F$ be a semi-normal filter of $L$. Then $F$ is a normal (fantastic) filter. Proof. Let $F$ be a semi-normal filter of $L$. Then by Theorems 5.4, $F$ is a MV-algebra and by Theorem 2.10, $F$ is a fantastic (normal) filter.

Solutions for two open problems in fantastic filters in BL-algebras

In [3], the definitions almost top elements and the set of double complemented elements were studied by A. Borumand Saeid (2009). In that paper there were two open problems for which the answer follows:

**Open problem 1.** [3] Under which one suitable conditions, if $F$ is a filter of $L$ such that $F = D(F)$, then $F$ is a fantastic filter?

**Open problem 2.** [3] Under which one suitable conditions, if $F$ is a filter of $L$ such that $(\overline{F}) = \{\{1\}\}$, then is a fantastic filter?

For more details, we review some related definitions and theorems.

**Definition 6.1.**[3] Let $F$ be a filter of $L$. Then (i) the set of double complemented elements, $D(F)$ is defined by $D(F) = \{x \in L \mid x^* \in F\}$.

(ii) An element $x \in L$ is called an almost top element of $L$, if $x^* = 1$.

(iii) We define $N(L)$ as follow: $N(L) = \{x \in L \mid x$ is an almost top element of $L\}$.
\( \{ x \in L \mid x^* = 1 \} = \{ x \in L \mid x^* = 0 \} \)

**Corollary 6.2.** \( D(\{1\}) = \{ x \in L \mid x^* = 1 \} \) and \( D(\{1\}) = \{ x \in L \mid x^* = 0 \} \).

**Corollary 6.3.** We can see \( N(L) \) is exactly \( ST_r \) of \( \{0\} \) and when \( N(L) = \{1\} \), \( L \) satisfies in \( ZRS \)-condition and so \( L \) is a MV-algebra if and only if \( N(L) = \{1\} \).

**Theorem 6.4.**[3] Let \( F \) be a filter of \( L \). Then

\( i) \) If \( F \) is a fantastic filter of \( L \), then \( F = D(F) \) and \( N(\frac{F}{F}) = \{1\} \).

\( ii) \) If \( F = D(F) \) and only if any almost top element of \( \frac{F}{F} \) is trivial.

**Theorem 6.5.** Let \( F \) be a filter of \( L \). Then

\( i) \) If \( F = D(F) \), then \( F \) is a fantastic filter.

\( ii) \) If \( N(\frac{F}{F}) = \{1\} \), then \( F \) is a fantastic filter.

**Proof.** (i) Since \( F = D(F) \), then by Theorem 6.4 (ii) any almost top element of \( \frac{F}{F} \) is trivial. Therefore, \( N(\frac{F}{F}) = \{1\} \) and so by Corollary 6.3, \( \frac{F}{F} \) is an MV-algebra. Hence, by Theorem 2.10, \( F \) is a fantastic filter.

(ii) By (i), the proof is clear.

**Conclusion**

In this paper we introduced the notion of left, right and product stabilizers in BL-algebras. By this notion we define BL-algebras with ZRS-condition such that they are equal to class MV-algebras and we establish that \( F \) is a normal filter if and only if \( F \) is a fantastic filter if and only if \( F = D(F) \). The results of this paper will be devoted to study the MV-algebra and Gödel algebra which are different extension of Basic Logic.

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**References**