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Super edge-magic total labeling in Extended Duplicate Graph of path

P.P. Ulaganathan, K. Thirusangu and B. Selvam
Department of Mathematics, S.I.V.E.T. College, Gowrivakkam, Chennai-600 073, India
ppulaganathan@gmail.com, kthirusangu@gmail.com, chellamaths@yahoo.com

Abstract
In this paper, we prove that the Extended Duplicate Graph of a path is super edge-magic

Keywords: Graph labeling, super edge-magic total labeling, Extended Duplicate Graphs
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Introduction

The concept of graph labeling was introduced by Rosa (1967). Graph labeling is an assignment of integers to the edges or vertices or both subject to certain conditions. Labeled graphs serve as useful models in a broad range of applications such as circuit design, communication network addressing, X-ray crystallography, radar, astronomy, data base management and coding theory. Over the past three decades various labeling of graphs such as cordial labeling, prime labeling, binary labeling, magic labeling, anti-magic labeling, bi-magic labeling, mean labeling, arithmetic labeling, graceful labeling, harmonious labeling etc., have been investigated in the literature (Gallian, 2010).

Kotzig and Rosa (1970) introduced an edge-magic total labeling. They proved $K_{m, n}$ has an edge-magic total labeling for all m and n and C_n has an edge-magic total labeling for all n $\geq$ 3. Enomoto *et al.* (1998) introduced super edge-magic total labeling. Wallis (2001) proved the following: C_n is super edge-magic if and only if n is odd; caterpillars are super edge-magic; $K_{m, n}$ is super edge-magic if and only if m is 1 or n = 1 and K_n is super edge-magic if and only if n = 1, 2 or 3. Figueroa-Centeno *et al.* (2001) proved that a graph is super edge-magic if and only if it is strongly 1 – harmonious and that a super edge-magic is cordial.

Though super edge-magic total labeling has been studied for different kinds of graphs, super edge-magic total labeling for duplicate graphs have not been investigated. In this paper we prove that Extended Duplicate Graph of path graphs are super edge-magic.

Preliminaries

In this section, we give the basic notions relevant to this paper. Let G = G (V, E) be a finite, simple and undirected graph. By a labeling we mean one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels (usually the set of integers).

Definition An edge-magic total labeling of a graph G (V, E) is a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant.

Definition An edge-magic total labeling of G (V, E) is called a Super edge-magic if $f(V) = \{1, 2, \ldots, |V|\}$ and $f(E) = \{|V| + 1, |V| + 2, \ldots, |V \cup E|\}$

Definition Let G (V, E) be a simple graph. A duplicate graph of G is DG = (V_1, E_1) where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $f: V \rightarrow V'$ is bijective (for $v \in V$, we write $f(v) = v'$ for convenience) and the edge set E_1 of DG is defined as: The edge ab is in E if and only if both ab' and $a'b$ are edges in E_1. Clearly the duplicate graph of the path graph is disconnected. We give the following definition from Thirusangu *et al.* (2010).

Definition Let DG = (V_1, E_1) be a duplicate graph of the path graph G (V, E). We add an edge between any one vertex from V to any other vertex in V′, except the terminal vertices of V and V′. For convenience, let us take $v_2 \in V$ and $v_2^1 \in V'$ and thus the edge (v_2, v_2^1) is formed. This graph is called the Extended Duplicate Graph of the path P_m and it is denoted by EDG (P_m).

Super edge-magic total labeling for EDG (P_m)

In this section, we present an algorithm and prove the existence of super edge-magic total labeling for EDG (P_m).

We assign the name to all edges in the following manner: In even paths, the edges (v_1, v_2'), (v_2', v_3), (v_3, v_4') ,..., (v_m', v_{m+1}) are named as e_1, e_2, e_3, ..., e_m; the edge (v_2, v_2') is named as e_{m+1} and the edges (v_1', v_2), (v_2, v_3'), (v_3', v_4) ,..., (v_m, v_{m+1}') are named as $e_1', e_2', e_3', \ldots, e_m'$ respectively.

In case of odd paths, the edges (v_1, v_2'), (v_2', v_3), (v_3, v_4') ,..., (v_m, v_{m+1}') are named as e_1, e_2, e_3, ..., e_m; the edge (v_2, v_2') is named as e_{m+1} and the edges (v_1', v_2), (v_2, v_3'), (v_3', v_4) ,..., (v_m', v_{m+1}) are named as $e_1', e_2', e_3', \ldots, e_m'$ respectively.

Algorithm

Input: EDG (P_m) with 2m+2 vertices 2m+1 edges.

Step 1: Denote the vertices of EDG (P_m) as $V = \{v_1, v_2, \ldots, v_m, v_{m+1}, v_1', v_2' \ldots v_m', v_{m+1}'\}$ and the edges as $E = \{e_1, e_2, e_3, \ldots, e_m, e_{m+1}, e_1', e_2', e_3', \ldots, e_m'\}$

Step 2: If 'm' is even, then m = 2n; $n \in$ N. The EDG (P_m) is of the form EDG (P_{2n}); $n \in$ N.

Define $f: V \rightarrow \{1,2,\ldots, |V|\}$ and
$f: E \rightarrow \{|V| + 1, |V| + 2, \ldots, |V \cup E|\}$

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For $1 \le i \le (m+2)/2$
$f(v_{2i-1}) = (m+2)-i$, $f(v'_{2i-1}) = (m+1)+i$;
For $1 \le i \le m/2$
$f(v_{2i}) = i$, $f(v'_{2i}) = (2m+3)-i$,
$f(e_{2i-1}) = (2m+1)+2i$, $f(e_{2i}) = (2m+2)+2i$,
$f(e'_{2i-1}) = (4m+5) - 2i$, $f(e'_{2i}) = (4m+4)-2i$
and $f(e_{m+1}) = 3m + 3$
Step 3: If 'm' is odd, then $m = 2n+1$; $n \in$ N. The EDG (P_m) is of the form EDG (P_{2n+1}); $n \in$ N.
Define $f: V \to \{1,2,\ldots, |V|\}$ and
$f: E \to \{|V| + 1, |V| + 2, \ldots, |V \cup E|\}$
For $1 \le i \le (m+1)/2$
$f(v_{2i-1}) = (m+2)-i$, $f(v'_{2i-1}) = (m+1)+i$,
$f(v_{2i}) = i$, $f(v'_{2i}) = (2m+3)-i$,
$f(e_{2i-1}) = (2m+1)+2i$, $f(e'_{2i-1}) = (4m+5) - 2i$;
For $1 \le i \le (m-1)/2$
$f(e_{2i}) = (2m+2)+2i$, $f(e'_{2i}) = (4m+4)-2i$
and $f(e_{m+1}) = 3m+3$
Step 4: Define $f^* : E \to N$ such that
$f^*(v_i, v_j) = f(v_i) + f(v_j) + f(v_i, v_j)$
Output: Labeled EDG (P_m)
Theorem 3.1: The Extended Duplicate Graph of path P_m, $m \ge 2$ admits super edge-magic total labeling.
Proof: Let EDG (P_m) be an Extended duplicate graph of path P_m. Clearly EDG (P_m) has 2m+2 vertices and 2m+1 edges.
Denote the set of vertices as V = (v_1, v_2, …, v_m, v_{m+1}, $v'_1, v'_2 \ldots v'_m, v'_{m+1}$}
Denote the set of edges as E = { e_1, e_2, e_3, …, e_m, e_{m+1}, $e'_1, e'_2, e'_3, \ldots, e'_m$}
Case 1: In this case, we prove the theorem for even paths. Let P_m be a path, where m=2n; $n \in$ N.
Consider the paths of the type P_{2n}; $n \in$ N.

To get an super edge-magic total labeling, define a map $f: V \to \{1, 2, \ldots, |V|\}$ and $f: E \to \{|V| + 1, |V| + 2, \ldots, |V \cup E|\}$ as given in step 2 of the above algorithm.

Therefore the vertices $v_2, v_4, v_6, \ldots, v_m, v_{m+1}, v_{m-1}, v_{m-3}, \ldots, v_1$ receives consecutive numbers such as 1, 2, …, m+1 and $v'_1, v'_3, v'_5 \ldots v'_{m+1}, v'_m, v'_{m-2}, v'_{m-4}, \ldots, v'_2$ receives consecutive numbers such as m+2, m+3, m+4, …, 2m+2.

The edges of $e_1, e_2, e_3, \ldots, e_m, e_{m+1}$ receive consecutive numbers such as 2m+3, 2m+4, 2m+5, …, 3m+2, 3m+3 and $e'_m, e'_{m-1}, e'_{m-2}, \ldots, e'_1$ receive consecutive numbers such as 3m+4, 3m+5, …, 4m+3.
Define the induced function $f^* : E \to N$ such that
$f^*(v_i, v_j) = f(v_i) + f(v_j) + f(v_i, v_j)$
From the definition of EDG (P_m), the (2m+1) edges of EDG (P_m) are in following five forms, namely (v_{2i-1}, v'_{2i}), (v'_{2i-1}, v_{2i}) where $1 \le i \le m/2$; (v_{2i}, v'_{2j-1}), (v'_{2i}, v_{2j-1}) where $j = i + 1$, $1 \le i \le m/2$ and (v_2, v'_2).
Now f^* (E) is computed as follows:
(i) For any i, where $1 \le i \le m/2$,
$f^*(v_{2i-1}, v'_{2i}) = f(v_{2i-1}) + f(v'_{2i}) + f(v_{2i-1}, v'_{2i})$
$= f(v_{2i-1}) + f(v'_{2i}) + f(e_{2i-1})$
$= \{(m+2)-i\} + \{(2m+3)-i\} + \{(2m+1)+2i\} = 5m + 6$
$f^*(v'_{2i-1}, v_{2i}) = f(v'_{2i-1}) + f(v_{2i}) + f(v'_{2i-1}, v_{2i})$
$= f(v'_{2i-1}) + f(v_{2i}) + f(e'_{2i-1})$
$= \{(m+1)+i\} + i + \{(4m+5)-2i\} = 5m + 6$
(ii) For any j, where $j = i+1$ and $1 \le i \le m/2$
$f^*(v_{2i}, v'_{2j-1}) = f(v_{2i}) + f(v'_{2j-1}) + f(v_{2i}, v'_{2j-1})$
$= f(v_{2i}) + f(v'_{2j-1}) + f(e'_{2i})$
$= i + \{(m+1)+j\} + \{(4m+4)-2i\} = 5m + 6$
$f^*(v'_{2i}, v_{2j-1}) = f(v'_{2i}) + f(v_{2j-1}) + f(v'_{2i}, v_{2j-1})$
$= f(v'_{2i}) + f(v_{2j-1}) + f(e_{2i})$
$= \{(2m+3)-i\} + \{(m+2)-j\} + \{(2m+2)+2i\} = 5m + 6$
(iii) For the edge (v_2, v'_2)
$f^*(v_2, v'_2) = f(v_2) + f(v'_2) + f(v_2, v'_2)$
$= f(v_2) + f(v'_2) + f(e_{m+1})$
$= 1+(2m+2)+(3m+3) = 5m + 6$

Thus f^* (E) = 5m+6 and hence EDG (P_m) admits super edge-magic total labeling with 5m + 6 as a magic constant.
Case (2): In this case, we prove the theorem for odd paths. Let P_m be a path, where $m = 2n+1$; $n \in$ N.
Consider the paths of the type P_{2n+1}; $n \in$ N.
To get super edge-magic total labeling, define a map $f: V \to \{1, 2, \ldots, |V|\}$ and $f: E \to \{|V| + 1, |V| + 2, \ldots, |V \cup E|\}$ as given in step 3 of the above algorithm.

Therefore the vertices $v_2, v_4, v_6, \ldots, v_m, v_{m+1}, v_{m-1}, v_{m-3}, \ldots, v_1$ receives consecutive numbers such as 1, 2, …, m+1 and $v'_1, v'_3, v'_5 \ldots v'_{m+1}, v'_m, v'_{m-2}, v'_{m-4}, \ldots, v'_2$ receives consecutive numbers such as m+2, m+3, m+4, …, 2m+2.

The edges of $e_1, e_2, e_3, \ldots, e_m, e_{m+1}$ receive consecutive numbers such as 2m+3, 2m+4, 2m+5, …, 3m+2, 3m+3 and $e'_m, e'_{m-1}, e'_{m-2}, \ldots, e'_1$ receive consecutive numbers such as 3m+4, 3m+5, …, 4m+3.
Define the induced function $f^* : E \to N$ such that
$f^*(v_i, v_j) = f(v_i) + f(v_j) + f(v_i, v_j)$
From the definition of EDG (P_m), the (2m+1) edges of EDG (P_m) are in the following five forms, namely (v_{2i-1}, v'_{2i}), (v'_{2i-1}, v_{2i}) where $1 \le i \le (m+1)/2$; (v_{2i}, v'_{2j-1}), (v'_{2i}, v_{2j-1}) where $j = i + 1$, $1 \le i \le (m-1)/2$ and (v_2, v'_2).

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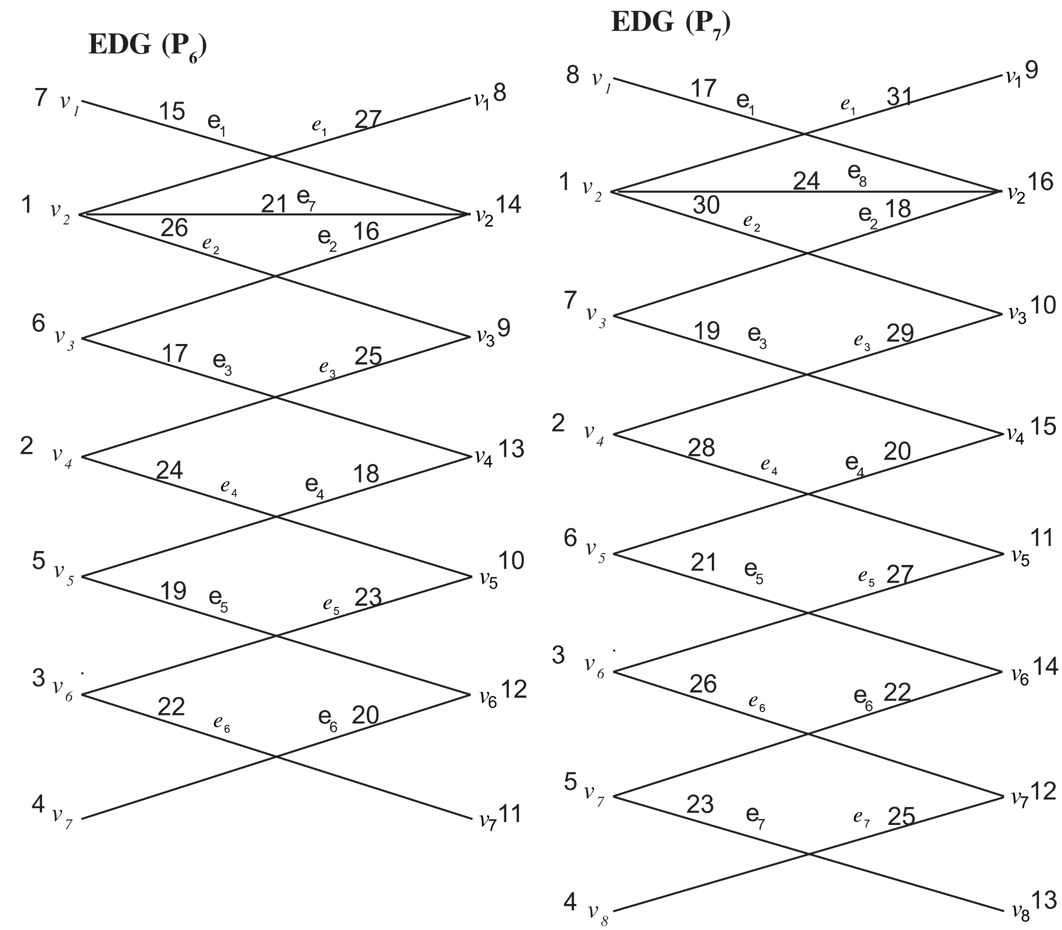
Fig. 1. Super-edge-magic total labeling



Now f^* (E) is computed as follows:
For any i, where $1 \le i \le (m+1)/2$,

(i) $f^*(v_{2i-1}, v'_{2i}) = f(v_{2i-1}) + f(v'_{2i}) + f(v_{2i-1}, v'_{2i})$

$= f(v_{2i-1}) + f(v'_{2i}) + f(e_{2i-1})$

$= \{(m+2)-i\} + \{(2m+3)-i\} + \{(2m+1)+2i\} = 5m + 6$

$f^*(v'_{2i-1}, v_{2i}) = f(v'_{2i-1}) + f(v_{2i}) + f(v'_{2i-1}, v_{2i})$

$= f(v'_{2i-1}) + f(v_{2i}) + f(e'_{2i-1})$

$= \{(m+1)+i\} + i + \{(4m+5)-2i\} = 5m + 6$

(ii) For any j, where j = i+1 and $1 \le i \le (m-1)/2$

$f^*(v_{2i}, v'_{2j-1}) = f(v_{2i}) + f(v'_{2j-1}) + f(v_{2i}, v'_{2j-1})$

$= f(v_{2i}) + f(v'_{2j-1}) + f(e'_{2i})$

$= i + \{(m+1)+j\} + \{(4m+4)-2i\} = 5m + 6$

$f^*(v'_{2i}, v_{2j-1}) = f(v'_{2i}) + f(v_{2j-1}) + f(v'_{2i}, v_{2j-1})$

$= f(v'_{2i}) + f(v_{2j-1}) + f(e_{2i})$

$= \{(2m+3)-i\} + \{(m+2)-j\} + \{(2m+2)+2i\} = 5m + 6$

(iii) For the edge (v_2, v'_2)

$f^*(v_2, v'_2) = f(v_2) + f(v'_2) + f(v_2, v'_2)$

$= f(v_2) + f(v'_2) + f(e_{m+1})$

$=1+(2m+2)+(3m+3)$

$= 5m + 6$

Thus f^* (E) = 5m+6 and hence EDG (P_m) admits super edge-magic total labeling with 5m + 6 as a magic constant.

Illustration: Super edge-magic total labeling for the graphs EDG (P_6) and EDG (P_7) are shown in the annexure.

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