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Felicitious labeling in Extended Duplicate Graph of Twig T_m

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Abstract

In this paper, we show that the class of Extended Duplicate Graph of a Twig is Felicitious.

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Introduction

The concept of graph labeling was introduced by Rosa (1967). A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as circuit design, radar, astronomy, coding theory, communication network addressing and data base management (Gallian, 2010). Hence, in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, mean labeling, arithmetic labeling etc., have been studied in over 1100 papers (Gallian, 2010).

The study of graceful graphs and graceful labeling methods was introduced by Rosa (1967). Rosa defined a β - valuation of a graph G with e edges as an injection from the vertices of G to the set {0,1,2,…..e} such that when each edge xy is assigned the label $\left|f(x)-f(y)\right|$ the resulting edge labels are distinct. β - valuations are functions that produce graceful labelings.

However, the term graceful labeling was not used until Golomb (1972) studied such labelings several years later.

An injective function from the vertices of a graph G with q edges to the set {0, 1, . . . , q} is called felicitious if the edge labels induced by {f(x) + f(y)} (mod q) for each edge xy are distinct. This definition first appeared in a paper by Lee *et al*. (1991) and is attributed to Balakrishnan and Kumar (1994) proved the conjecture of Lee *et al*. (1991) that every graph is a subgraph of a felicitious graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitious are: C_n except when n = 2 (mod 4); $K_{m,n}$ when m, n >1; Lee *et al*. (1991) conjectured that the n-cube is felicitious. Balakrishnan *et al*. (1996) obtained numerous results on felicitious labelings. Yegnanarayanan (1999) conjectures that the graphs obtained from an even cycle by attaching n new vertices to each vertex of the cycle is felicitious. In this paper, we prove that the Extended Duplicate Graph of Twigs are Felicitious.

Preliminaries

In this section we give the basic notions relevant to this paper. Let *G = G (V,E)* be a finite, simple and undirected graph with *p* vertices and *q* edges. By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels. In this paper we deal labeling with domain as the set of all vertices.

Definition (Graceful) : A function f is said to be Graceful of a graph G with 'q' edges if f is 1-1 from V→{0,1,2,3….q} such that for each edge uv is assigned the label $\left|f(u)-f(v)\right|$, the resulting edge labels are distinct numbers {1,2,3,…q}.

Definition (Felicitious) : An injective function $f: V \rightarrow \{0,1,2.....q\}$ where $\left|E\right| = q$, is called felicitious, if the edge labels induced by {f(x) + f(y)} (mod q) for each edge xy are distinct.

Definition (Twig) : A graph *G(V,E)* obtained from a path by attached exactly two pendent edges to each internal vertices of the path is called a Twig graph. Generally, a Twig T_m with *m* internal vertices, has 3m+1 edges and 3m+2 vertices. We classify the Twigs into two categories such as T_{2n} and T_{2n-1}, $n \in$ N.

Definition Let *G (V,E)* be a simple graph. A duplicate graph of *G* is $DG = (V_1, E_1)$ where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $f : V \rightarrow V'$ is bijective, (for $v \in V$, we write $f(v)$ = v' for convenience) and the edge set E_1 of *DG* is defined as follows: The edge *ab* is in *E* if and only if both ab' and a'b are edges in E_1. Clearly the duplicate graph of the twig graph is disconnected. In order to view as a connected graph we give the following definition from Selvam *et al.* (2010, 2011).

Definition EDG (T_m): Let *G (V,E)* be a Twig graph T_m and let $DG = (V_1, E_1)$ where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ be a duplicate graph of a Twig T_m. Add an edge between any one vertex from *V* to any other vertex in *V'*, except the terminal vertices of *V* and *V'*. For convenience, let us take $v_2 \in V$ and $v_2' \in V'$ and thus the edge (v_2, v_2') is formed. We call this new graph as the

Extended Duplicate Graph of the Twig T_m and it is denoted as *EDG (T_m).* Clearly this EDG (T_m) has *(6m+4)* vertices and *(6m+3)* edges.

Results

In this section, we present an algorithm and prove the existence of felicitious labeling for EDG(T_m).

Algorithm:

Input: EDG (T_m) with 6m+4 vertices and 6m+3 edges.

Step 1 : Denote the 6m+4 vertices as

V= $\left\{v_1, v_2 \ldots, v_{3m+1}, v_{3m+2}, v_1^{'}, v_2^{'} \ldots, v_{3m+1}^{'}, v_{3m+2}^{'}\right\}$

Step 2 : If 'm' is even, then m = 2n ; n$\in$N

If n$\in$N, then the EDG (T_m) is of the form EDG (T_{2n}); n$\in$N

Define f: V $\rightarrow$ { 0,1,2,3….6m+3} such that, for $0 \le i \le \dfrac{m-2}{2}$

(i). $f\left(v_{6i+3}\right) = 3(m-i)$, $f\left(v_{6i+4}\right) = 3(m-i)-1$, $f\left(v_{6i+5}\right) = 3(m-i)-2$

(ii). $f\left(v_{6i+6)}\right) = 3i+1,\ f\left(v_{6i+7}\right) = 3i+2,\ f\left(v_{6i+8}\right) = 3i+3$

(iii). $f\left(v_{6i+3}^{'}\right) = 3(m+i)+3,\ f\left(v_{6i+4}^{'}\right) = 3(m+i)+4,\ f\left(v_{6i+5}^{'}\right) = 3(m+i)+5$

(iv). $f\left(v_{6i+6}^{'}\right) = 3(2m-i)+2,\ f\left(v_{6i+7}^{'}\right) = 3(2m-i)+1,\ f\left(v_{6i+8}^{'}\right) = 3(2m-i)$

and also

$f\left(v_1\right) = 3m+1,\ f\left(v_2\right) = 0,\ f\left(v_1^{'}\right) = 3m+2,\ f\left(v_2^{'}\right) = 6m+3$

Step 3:

If m is odd, then m=2n-1; n$\in$N

The EDG(T_m) is of the form EDG (T_{2n-1}); n$\in$N

Define f: V$\rightarrow${ 0,1,2,3….6m+3} such that $f\left(v_i\right)$ is same as in (i) to (iv) of **Step 2** with the limits $0 \le i \le \dfrac{m-1}{2}$, $0 \le i \le \dfrac{m-3}{2}$, $0 \le i \le \dfrac{m-1}{2}$ and $0 \le i \le \dfrac{m-3}{2}$ respectively.

Step 4:

Define $f^*: E \rightarrow N$ such that

$f^*\left(v_i, v_j\right) = \{f(v_i) + f(v_j)\}(\mathrm{mod}\, q)$

Output : Labeled EDG(T_m)

Theorem:

The Extended Duplicate Graph of Twig T_m, m$\ge$ 1 is felicitious.

Proof:

Let EDG (T_m) be a Extended duplicate graph of the Twig T_m. Clearly

EDG(T_m) has 6m+4 vertices and 6m+3 edges.

Denote the set of vertices as

V= $\left\{v_1, v_2 \ldots, v_{3m+1}, v_{3m+2}, v_1^{'}, v_2^{'} \ldots, v_{3m+1}^{'}, v_{3m+2}^{'}\right\}$

Case (1)

Let T_m be a Twig where m=2n, n$\in$N.

Consider the Twig of the Type T_{2n}, n$\in$N.

To label the vertices, define a map f:V$\rightarrow${0,1,2,3….6m+3} as given in **Step 2** of the above algorithm.

The vertices v_1 and v_2 are labeled as 3m+1 and 0; the vertices (v_3, v_4, v_5), (v_9, v_{10}, v_{11}) …………. $(v_{3m-3}, v_{3m-2}, v_{3m-1})$, $(v_{3m+2}, v_{3m+1}, v_{3m})$, $(v_{3m-4}, v_{3m-5}, v_{3m-6})$….. (v_8, v_7, v_6) receive consecutive numbers such as 3m, 3m-1, 3m-2,…. 3,2,1 as labels; the vertices $v_1^{'}$ and $v_2^{'}$ are labeled as 3m+2 and 6m+3; the vertices $(v_3^{'}, v_4^{'}, v_5^{'})$, $(v_9^{'}, v_{10}^{'}, v_{11}^{'})$ …………. $(v_{3m-3}^{'}, v_{3m-2}^{'}, v_{3m-1}^{'})$, $(v_{3m+2}^{'}, v_{3m+1}^{'}, v_{3m}^{'})$, $(v_{3m-4}^{'}, v_{3m-5}^{'}, v_{3m-6}^{'})$….. $(v_8^{'}, v_7^{'}, v_6^{'})$ receive consecutive numbers 3m+3, 3m+4, 3m+5,…… 6m, 6m+1, 6m+2 as labels.

Thus all the 6m+4 vertices are labeled.

From the definition of EDG(T_m), the 6m+3 edges of EDG(T_m) are of the form $\left(v_2, v_2^{'}\right)$, $\left(v_1, v_2^{'}\right)$, $\left(v_2, v_1^{'}\right)$, $\left(v_{2+3i}, v_{3i+3+j}^{'}\right)$, $\left(v_{2+3i}^{'}, v_{3i+3+j}\right)$ for $0 \le j \le 2,\ 0 \le i \le m-1$.

The induced function $f^*: E \rightarrow N$ is defined such that

$f^*\left(v_i, v_j\right) = \{f(v_i) + f(v_j)\}(\mathrm{mod}\, q)$

Thus the edges $\left(v_1^{'}, v_2\right)$, $\left(v_2, v_3^{'}\right)$, $\left(v_2, v_4^{'}\right)$, $\left(v_2, v_5^{'}\right)$, $\left(v_5^{'}, v_6\right)$, $\left(v_5^{'}, v_7\right)$, $\left(v_5^{'}, v_8\right)$ ….. $\left(v_{3m-1}^{'}, v_{3m}\right)$, $\left(v_{3m-1}^{'}, v_{3m+1}\right)$, $\left(v_{3m-1}^{'}, v_{3m+2}\right)$ receive consecutive integers such as 3m+2, 3m+3, 3m+4,3m+5……6m, 6m+1, 6m+2 as labels ; the edge $\left(v_2, v_2^{'}\right)$ receive the number 0 as label; the edges $\left(v_{3m+2}^{'}, v_{3m-1}\right)$, $\left(v_{3m+1}^{'}, v_{3m-1}\right)$, $\left(v_{3m}^{'}, v_{3m-1}\right)$,

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Fig. 1. Felicitous labeling EDG T_6

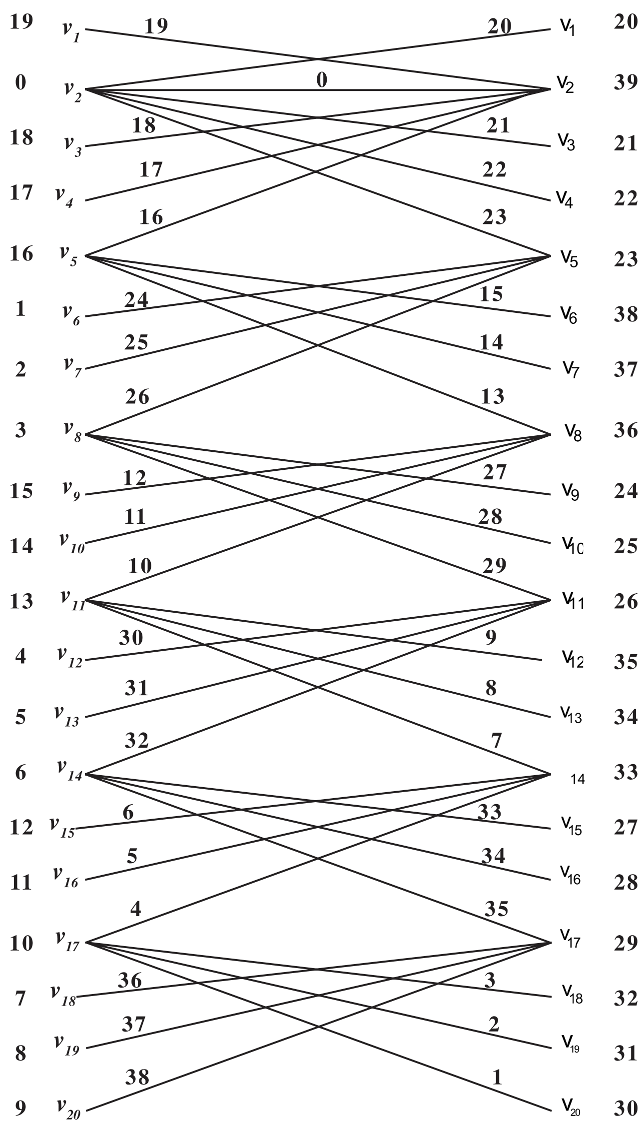
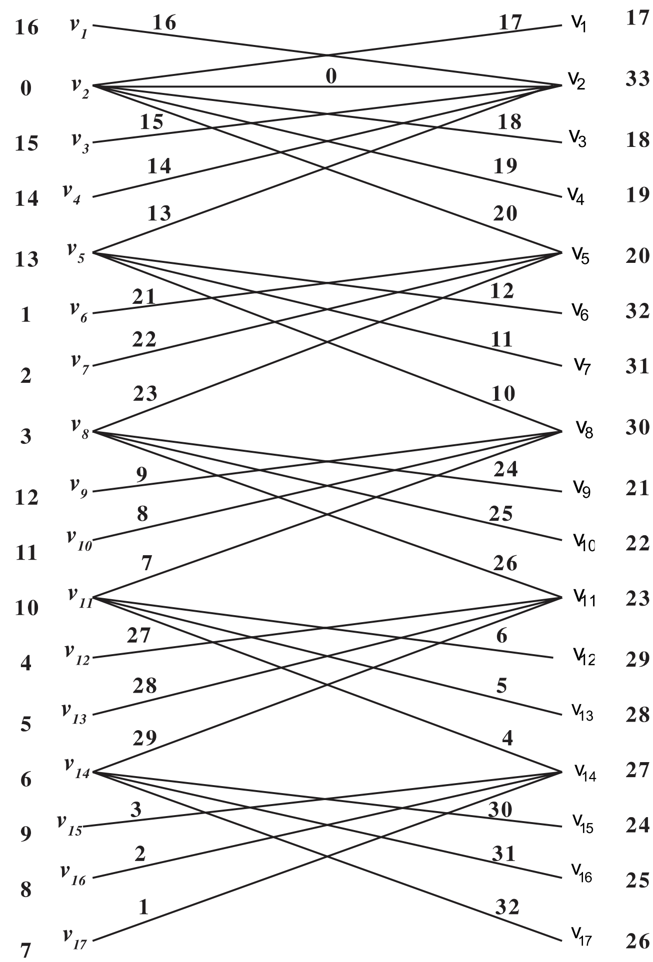


Fig. 2. Felicitous labeling EDG T_5



(v_{3m-1}, v'_{3m-4}), (v_{3m-2}, v'_{3m-4}), (v_{3m-3}, v'_{3m-4}).... (v_5, v'_2), (v_4, v'_2), (v_3, v'_2), (v'_2, v_1) receive consecutive integers such as 1,2,3,……3m-2, 3m-1, 3m, 3m+1 as labels.

Thus all the edges are labeled as {0,1,2,3,4,……6m+2} and all are distinct. (The whole idea of the proof is illustrated in Fig(2)).

Hence EDG(T_m) where m=2n , n$\in$N is felicitious.

Case 2:

Let T_m be a Twig where m=2n-1, n$\in$N.

Consider the Twig of the Type T_{2n-1}, n$\in$N.

To label the vertices, define a map f:V$\rightarrow${0,1,2,3….6m+3} as given in **Step 3** of the above algorithm.

The vertices v_1 and v_2 are labeled as 3m+1 and 0; the vertices (v_3, v_4, v_5) , (v_9, v_{10}, v_{11}) ………… $(v_{3m}, v_{3m+1}, v_{3m+2})$, $(v_{3m-1}, v_{3m-2}, v_{3m-3})$…… (v_8, v_7, v_6) receive consecutive integers such as 3m, 3m-1, 3m-2,…….3,2,1 as labels; the vertices v'_1 and v'_2 are labeled as 3m+2 and 6m+3; the vertices (v'_3, v'_4, v'_5) , (v'_9, v'_{10}, v'_{11})………… $(v'_{3m}, v'_{3m+1}, v'_{3m+2})$,

$(v'_{3m-1}, v'_{3m-2}, v'_{3m-3})$,…… (v'_8, v'_7, v'_6) receive consecutive integers such as 3m+3, 3m+4, 3m+5,…… 6m, 6m+1, 6m+2 as labels.

Thus all the 6m+4 vertices are labeled.

From the definition of EDG (T_m), the 6m+3 edges of EDG (T_m) are of the form (v_2, v_2'), (v_1, v_2'), (v_2, v_1'), (v_{2+3i}, v'_{3i+3+j}), (v'_{2+3i}, v_{3i+3+j}) for $0 \le j \le 2,\ 0 \le i \le m-1$.

The induced function $f^*: E \to N$ is defined such that

$$f^*(v_i, v_j) = \{f(v_i) + f(v_j)\}(\mathrm{mod}\, q)$$

Thus the edges (v_1', v_2), (v_2, v_3'), (v_2, v_4'), (v_2, v_5'), (v_5', v_6), (v_5', v_7), (v_5', v_8) ..... (v'_{3m-1}, v_{3m}), (v'_{3m-1}, v_{3m+1}), (v'_{3m-1}, v_{3m+2}) receive consecutive integers such as 3m+2, 3m+3, 3m+4,3m+5......6m, 6m+1, 6m+2 as labels ; the edge (v_2, v_2') receive the number 0 as label; the edges (v'_{3m+2}, v_{3m-1}), (v'_{3m+1}, v_{3m-1}), (v'_{3m}, v_{3m-1}), (v_{3m-1}, v'_{3m-4}), (v_{3m-2}, v'_{3m-4}), (v_{3m-3}, v'_{3m-4}) .... (v_5, v_2'), (v_4, v_2'), (v_3, v_2'), (v_2', v_1) receive consecutive integers such as 1,2,3,......3m-2, 3m-1, 3m, 3m+1 as labels.

Thus all the edges are labeled as {0,1,2,3,4,......6m+2} and all are distinct. (The above concept is illustrated in Fig(1)).

Hence EDG (T_m) where m=2n , n$\in$N is felicitious.

Hence the proof of the theorem is completed.

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