Indian Journal of Science and Technology Vol. 3 No. 3 (Mar 2010) ISSN: 0974- 6846

Periodic loop solutions of the CH-DP equation

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Abstract

Periodic loop solutions of the CH-DP equation are investigated by using the dynamical system theory. The solutions are characterized by two parameters. The periodic loops existent conditions are found. Explicit analytical periodic loop solutions of CH-DP equation are derived.

Keywords: Cammasa-Holm equation, CH-DP equation, traveling wave system, singular point, periodic loops.

Introduction

It is well known that the Cammasa-Holm (CH) equation (Camassa *et al.,* 1993):

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad (1.1)$$

is a model for unidirectional nonlinear dispersive waves in shallow water. This eqn. has attracted a lot of attention over the past decade due to its interesting mathematical properties, e.g., it is an integrable equation and admits a peakon solution. For k = 0, Camassa and Holm (1993) showed that Eqn. (1.1) has peaked solitary wave solutions.

A new variant of (1.1) was introduced by Degasperis and Procesi (1999):

$$m_t + um_x + bu_x m = c_0 u_x - \gamma u_{xxx} \quad (1.2)$$

Which is called the CH-DP equation, where $m = u - \alpha^2 u_{xx}$ is a momentum variable, α, c_0, b, γ are constants and $\alpha \neq 0$

In recent years, much work on the CH-DP equation has been done (Degasperis *et al.,* 1999; Degasperis *et al.,* 2002; Holm *et al.,* 2003; Dullin *et al.,* 2004; Vakhnenko *et al.,* 2004; Liu *et al.,* 2007; Li *et al.,* 2009; Xie *et al.,* 2010).

For c_0=0, γ =0, α =1 and b=3, Eqn. (1.2) becomes Degasperis-Procesi (DP) eqn.

$$u_t - c_0 u_x + 4uu_x = u_{xxt} + uu_{xxx} + 3u_x u_{xx} \quad (1.3)$$

Vakhnenko and Parkes (2004) have obtained that hump-like, loop-like and coshoidal periodic-wave solutions of Eqn. (1.3). The some exact travelling wave solutions of Eqn. (1.3) have been given by Li and Yi (2009). Xie and Wang (2010) obtained exact explicit parameter expressions of compactons and implicit expressions of generalized kink wave solutions of Eqn. (1.2) for b=3.

In this paper, for b=3, we to continue study the periodic loop wave solutions of Eqn. (1.2) by use of the dynamical system theory.

Properties of singular points

For b = 3 and $\alpha \neq 0$, substituting $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ in Eqn. (1.2), we have

$$-(c - c_0)\phi' + 4\phi\phi' - \alpha^2(\phi\phi''' + 3\phi'\phi'') + (\alpha^2 c + \gamma)\phi''' = 0 \quad (2.1)$$

where c is the wave speed.

Integrating (2.1) once with respect to ξ. We have the following travelling wave eqn.

$$-(c - c_0)\phi + 2\phi^2 - (\alpha^2\phi - \alpha^2 c - \gamma)\phi'' - \alpha^2(\phi')^2 + g = 0 \quad (2.2)$$

where g is integral constant.

Letting $\phi' = y$, we obtain a planar system

$$\begin{cases} \dfrac{d\phi}{d\xi} = y \\ \dfrac{dy}{d\xi} = \dfrac{-(c + c_0)\phi + 2\phi^2 - \alpha^2 y^2 + g}{\alpha^2\phi - \alpha^2 c - \gamma} \end{cases} \quad (2.3)$$

which is called travelling wave system. The system (2.3) has a singular line $\phi = q = c + \dfrac{\gamma}{\alpha^2}$ which is inconvenient to our study. So we make the transformation

$$d\xi = \alpha^2(\phi - q)d\tau \quad (2.4)$$

where τ is a parametric variable. Thus system (2.3) becomes

$$\begin{cases} \dfrac{d\phi}{d\xi} = \alpha^2(\phi - q)y \\ \dfrac{dy}{d\xi} = -(c + c_0)\phi + 2\phi^2 - \alpha^2 y^2 + g \end{cases} \quad (2.5)$$

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Obviously, systems (2.3) and (2.5) have the same first integral as follows:

$$H(\phi, y) = \alpha^2(\phi - q)y^2 - \phi^4 + \frac{2}{3}(c + c_0 +$$

$$2q)\phi^3 - ((c + c_0)q + g)\phi^2 + 2qg\phi = h \qquad (2.6)$$

Therefore both systems (2.3) and (2.5) have the same topological phase portraits except the straight line $\phi = q$.

Let

$$g_1(c) = \frac{(c + c_0)^2}{8} \qquad (2.7)$$

$$g_2(c) = \frac{(\alpha^2 c_0 - \gamma)(3\alpha^2 c + \alpha^2 c_0 + 2\gamma)}{9\alpha^4} \qquad (2.8)$$

and

$$g_3(c) = -\frac{(\alpha^2 c_0 + \gamma)(\alpha^2 c - \alpha^2 c_0 + 2\gamma)}{\alpha^4} \qquad (2.9)$$

It is easy to know that $g_3(c) \leq g_2(c) \leq g_1(c)$, and the ($c^*$, g^*) is unique intersection point of the three curves $g_1(c)$, $g_2(c)$ and $g_3(c)$, where $c^* = \frac{\alpha^2 c_0 - 4\gamma}{3\alpha^2}$ and $g^* = \frac{2(\alpha^2 c_0 - \gamma)}{9\alpha^4}$

Let

$$\phi_0 = \frac{c + c_0}{4} \qquad (2.10)$$

$$\phi_\pm = \frac{c + c_0 \pm \sqrt{(c + c_0)^2 - 8g}}{4}$$

for $g < g_1$ (2.11)

and

$$y_\pm = \pm\sqrt{\frac{1}{\alpha^2}(c^2 - (c_0 - \frac{3\gamma}{\alpha^2})c - \qquad \frac{\alpha^2 c_0\gamma - 2\gamma^2}{\alpha^4} + g)}$$

(Should be $y_\pm = \pm\sqrt{\frac{1}{\alpha^2}(c^2 - (c_0 - \frac{3\gamma}{\alpha^2})c - \frac{\alpha^2 c_0\gamma - 2\gamma^2}{\alpha^4} + g)}$) for $g > g_3$ (2.12)

Using the bifurcation method of planar systems, we know that the singular points of systems (2.5) have following properties.

(1). When $g > g_1$, $(q, y_\pm)$ are 2 saddle points.

(2). When g = g_1 and $c \neq c^*$, $(q, y_\pm)$ are 2 saddle points, $(\phi_0, 0)$ is a degenerate saddle point.

(3). When $g = g^*$ and $c = c^*$, (c^*, g^*) is a saddle point.

(4). When $g_3 < g < g_1$ and $c^* < c$, $(q, y_\pm)$ and $(\phi_-, 0)$ are 3 saddle points, $(\phi_+, 0)$ is a center point, and $H(q, y_\pm) = H(\phi_-, 0)$ when $g = g_2$.

(5). When $g = g_3$ and $c^* < c$, $(q, 0) = (\phi_+, 0)$ is a degenerate saddle point, $(\phi_-, 0)$ is saddle point.

(6). When $g < g_3$, $(\phi_\pm, 0)$ are 2 saddle points.

(7). When g = g_3 and c < c^*, $(q, 0) = (\phi_-, 0)$ is a degenerate saddle point, $(\phi_+, 0)$ is a saddle point.

(8). When $g_3 < g < g_1$ and c < c^*, $(q, y_\pm)$ and $(\phi_+, 0)$ are 3 saddle points, $(\phi_-, 0)$ is a center point, and $H(q, y_\pm) = H(\phi_+, 0)$ when $g = g_2$.

Periodic loop solutions

We assume that $(\phi_0^*, 0)$ is the initial point of a orbit of system (2.5), from (2.6), it has expression $H(\phi, y) = h_0$, where $h_0 = H(\phi_0^*, 0)$. Let

$$f(\phi) = \phi^4 - \frac{2}{3}(c + c_0 + 2q)\phi^3 +$$

$$((c + c_0)q + g)\phi^2 - 2qg\phi + h \qquad (3.1)$$

then the orbit of passing through $(\phi_0^*, 0)$ has expression

$$y = \pm\sqrt{\frac{f(\phi)}{\alpha^2(\phi - q)^2}} \qquad (3.2)$$

From (3.1), we have

$$f'(\phi) = 4\phi^3 - 2(c + c_0 + 2q)\phi^2 +$$

$$2((c + c_0)q + g)\phi - 2qg \qquad (3.3)$$

Obviously, the ϕ_-, q and ϕ_+ are 3 real roots of the $f'(\phi) = 0$ when $g < g_3$, where $\phi_- < q < \phi_+$. So the polynomial of fourth order $f(\phi)$ has three possible cases is shown in Fig.1.

Case 1. $c \geq c^*$, $g < g_3$ and $q < \phi_0^* < \phi_+$

In this case, ϕ_1, ϕ_2, ϕ_0^* and ϕ_3 are 4 real roots of $f(\phi) = 0$ (see Fig.1 (1-1 and 2)), (3.1) becomes

$$f(\phi) = (\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_0^*)(\phi - \phi_3) \qquad (3.4)$$

Thus (3.2) becomes

$$y = \pm\sqrt{\frac{(\phi_3 - \phi)(\phi_0^* - \phi)(\phi - \phi_2)(\phi - \phi_1)}{\alpha^2(\phi - q)^2}} \qquad (3.5)$$

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By using formula 255.00 (Byrd & Friedman, 1971), substituting (3.5) into the first expression of (2.5) and integrating it along interval $[\phi, \phi_0^*]$, we get

$$\phi = \frac{\phi_0^* - \phi_3 n_1^2 sn^2(w,k)}{1 - n_1^2 sn^2(w,k)} \tag{3.6}$$

where $w = \frac{|\alpha| \sqrt{(\phi_3 - \phi_2)(\phi_0^* - \phi_1)}}{2}\tau$ is a parameter variable, $k = \sqrt{\frac{(\phi_0^* - \phi_2)(\phi_3 - \phi_1)}{(\phi_3 - \phi_2)(\phi_0^* - \phi_1)}}$ is the modulus of Jacobian elliptic function $n_1 = \sqrt{\frac{\phi_0^* - \phi_2}{\phi_3 - \phi_2}}$, and

$\phi_2 \le \phi \le \phi_0^*$

By using formula 400.01(Byrd & Friedman, 1971) Substituting (3.6) into (2.4) and integrating it, we get

$$\xi = \frac{2|\alpha|}{\sqrt{(\phi_3 - \phi_2)(\phi_0^* - \phi_1)}}((\phi_3 - q)w - (\phi_3 - \phi_0^*)\prod(\arcsin(sn(w,k), n_1^2, k)), \tag{3.7}$$

Thus we obtain a periodic loop solution $u(x, t) = \phi(\xi)$ of parametric type as follows:

$$\begin{cases} \phi = \dfrac{\phi_0^* - \phi_3 n_1^2 sn^2(w,k)}{1 - n_1^2 sn^2(w,k)} \\ \xi = \dfrac{2|\alpha|}{\sqrt{(\phi_3 - \phi_2)(\phi_0^* - \phi_1)}}((\phi_3 - q)w - \\ \quad (\phi_3 - \phi_0^*)\prod(\arcsin(sn(w,k), n_1^2, k) \end{cases} \tag{3.8}$$

where $-\infty < \xi < +\infty$ and $\phi_2 \le \phi \le \phi_0^*$

Ex.1. When ϕ = 1, γ =1 and c_0=1, then c^*=-1. Choosing c =2, then g_1=1.125, g_2=0 and g =-9. Choosing g=-20, then q=3, ϕ_- =-2.5 and ϕ_+ =4. We take ϕ_0^* =3.55, then $\phi_1 \approx$ -4.467647305, $\phi_2 \approx$ 2.589713343 and $\phi_3 \approx$4.327933953. From (3.8) we can simulate a periodic loop of Eqn. (1.2) as (2-1) in Fig.2.

Ex.2. When ϕ =1, γ =1 and c_0=1, then c^*=-1. Choosing c=-1, then g_1=g_2=g_3=0. Choosing g =-15, then q=0, $\phi_- \approx$-2.738612788 and $\phi_+ \approx$ 2.738612788. We take ϕ_0^* =0.5, then $\phi_1 \approx$-3.840572874, $\phi_2 \approx$-0.5 and $\phi_3 \approx$3.840572874. From (3.8) we can simulate a periodic loop of Eqn. (1.2) as (2-2) in Fig.2.

Case 2. $c < c^*$, $g < g_3$ and $\phi_- < \phi_0^* < q$

In this case, ϕ_1, ϕ_0^*, ϕ_2 and ϕ_3 are 4 real roots of $f(\phi)$ = 0 (see Fig.1 (1-3)), (3.1) becomes

$$f(\phi) = (\phi - \phi_1)(\phi - \phi_0^*)(\phi - \phi_2)(\phi - \phi_3) \tag{3.9}$$

Thus (3.2) becomes

$$y = \pm\sqrt{\frac{(\phi_3 - \phi)(\phi_2 - \phi)(\phi - \phi_0^*)(\phi - \phi_1)}{\alpha^2(\phi - q)^2}} \tag{3.10}$$

By using formula 254.00 (Byrd & Friedman, 1971), substituting (3.10) into the first expression of (2.5) and integrating it along interval $[\phi_0^*, \phi]$, we get

$$\phi = \frac{\phi_0^* - \phi_1 n_2^2 sn^2(w,k)}{1 - n_2^2 sn^2(w,k)} \tag{3.11}$$

where $w = \frac{|\alpha| \sqrt{(\phi_3 - \phi_0^*)(\phi_2 - \phi_1)}}{2}\tau$ is a parameter variable, $k = \sqrt{\frac{(\phi_2 - \phi_0^*)(\phi_3 - \phi_1)}{(\phi_3 - \phi_0^*)(\phi_2 - \phi_1)}}$ is the modulus of Jacobian elliptic function $n_2 = \sqrt{\frac{\phi_2 - \phi_0^*}{\phi_2 - \phi_1}}$, and

$\phi_0^* \le \phi \le \phi_2$

By using formula 400.01 (Byrd & Friedman, 1971), substituting (3.11) into (2.4) and integrating it, we get

$$\xi = \frac{2|\alpha|}{\sqrt{(\phi_3 - \phi_0^*)(\phi_2 - \phi_1)}}((\phi_1 - q)w + (\phi_0^* - \phi_1)\prod(\arcsin(sn(w,k), n_2^2, k)) \tag{3.12}$$

Thus we obtain other solution $u(x, t) = \phi(\xi)$ of parametric type as follows:

$$\begin{cases} \phi = \dfrac{\phi_0^* - \phi_1 n_2^2 sn^2(w,k)}{1 - n_2^2 sn^2(w,k)} \\ \xi = \dfrac{2|\alpha|}{\sqrt{(\phi_3 - \phi_0^*)(\phi_2 - \phi_1)}}((\phi_1 - q)w + \\ \quad (\phi_0^* - \phi_1)\prod(\arcsin(sn(w,k), n_2^2, k) \end{cases} \tag{3.13}$$

where $-\infty < \xi < +\infty$ and $\phi_0^* \le \phi \le \phi_2$

Ex.3. When ϕ =1, γ =1 and c_0=1, then c^*=-1. Choosing c =-2, then g_1=0.125, g_2=0 and g_3= -1. Choosing g=-5, then q=-1, $\phi_- \approx$-1.850781059 and $\phi_+ \approx$1.350781059

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. We take $\phi_0^* = -1.57$, then $\phi_1 \approx -2.068455201$, $\phi_2 \approx -0.5797587044$ and $\phi_3 \approx 2.218213905$. From (3.8) we can simulate a periodic loop of Eqn. (1.2) as (2-3) in Fig.2. From Fig.2, it is shown that periodic loops are inversion for $c > c^*$ and $c < c^*$ (see Fig.2 (2-1 and 3)), the loop is symmetry on ξ -- axes for $c = c^*$ (see Fig.2 (2-2)).

Conclusion

In this paper, we have studied the bifurcation and global behavior a CH-DP eqn. and obtained the conditions under which the periodic loop waves appear and their representations be obtained. Their planar graphs are simulated under the some parameter (see Fig.2). These results are new to CH-DP equation.

Fig.1. Three possible cases for the polynomial $f(\phi)$ when $g < g_3$. (1-1). $c > c^*$. (1-2). $c = c^*$. (1-3). $c < c^*$}

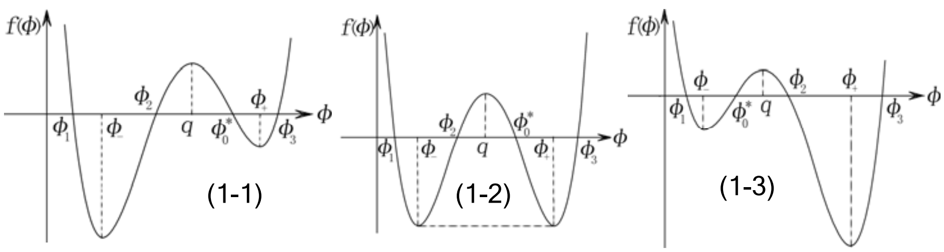
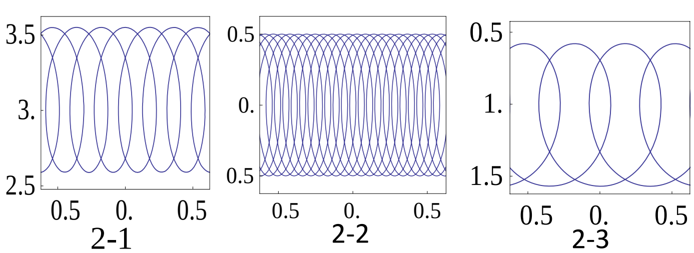


Fig.2. The periodic loops of Eqn. (1.2) with α =1, γ =1 and c_0=1. (2-1). c=2, g=-20 and ϕ_0^*=3.55. (2-2). c=-1, g=-15 and ϕ_0^*=0.5. (2-3). c=-2, g=-5 and ϕ_0^*=-1.57



Acknowledgement

The authors would like to express their sincere thanks to the reviewer for suggestions and comments. This work was supported by the Natural Science Foundation of China (No.10571062) and Yunnan Province (2008ZC153M).

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