A general class of multivariate distribution involving $\overline{H}$-function

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Abstract: In this paper an attempt has been made to present unified theory of the classical statistical distribution associated with the multivariate generalized Dirichlet distribution involving $\overline{H}$-function with general arguments. In particular, Mathematical expectation of a general class of polynomials, characteristic function and the distribution function are investigated.
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Introduction

The $\overline{H}$-function occurring in the paper will be defined and represented as follows:

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=\overline{H}_{P,Q}^{M,N}\left[z\,\Big|^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \qquad (1.1)$$

Where

$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (1.2)$$

For further details of $\overline{H}$-function, the original paper of Buschman and Srivastava (1990) was referred.

The general class of polynomials defined by Srivastava (1972) is represented in the following manner:

$$S_n^m\left[x\right]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k},n=0,1,2... \qquad (1.3)$$

Where m is an arbitrary positive integer and the coefficients $A_{n,k}(n,k\geq 0)$ are arbitrary constant, real or complex.

We shall use the following notation:

$$A^*=\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P};B^*=(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}$$

Probability density functions

This paper deals with certain classical statistical distributions associated with Drichlet distributions or multivariate analogue of the beta distribution. The probability density function is taken in terms of h-function defined by Buschman and Srivastava (1990) with general arguments.

Let

$$f(x_1,...,x_k)=K\left(1-\sum_{j=1}^{k}C_jx_j^{t_j}\right)^{\rho-1}\prod_{j=1}^{k}\left(x_j\right)^{s_j-1}\overline{H}_{P,Q}^{M,N}\left[a\left(1-\sum_{j=1}^{k}C_jx_j^{t_j}\right)^{\sigma}\prod_{j=1}^{k}\left(x_j\right)^{U_j}\right] \qquad (2.1)$$

At any point of the region R defined by

$x_j\geq 0\left(j=1,...,k\right)$ *and* $\sum_{j=1}^{k}C_jx_j^{t_j}\leq 1$ *and* $f(x_1,...,x_k)=0$, out side the region. Also

$$K^{-1}=\prod_{j=1}^{k}\left\{\frac{C^{-(s_j/t_j)}}{t_j}\right\}\overline{H}_{P+k+1,Q+1}^{M,N+k+1}\left[a\sum_{j=1}^{k}(C_j)^{-(U_j/t_j)}\,\Bigg|^{\left(1-\frac{s_j}{t_j},\frac{U_j}{t_j};1\right)_{1,k},(1-\rho,\sigma;1),A^*}_{B^*,\left(1-\rho-\sum_{j=1}^{k}\frac{s_j}{t_j},\sum_{j=1}^{k}\frac{U_j}{t_j}+\sigma;1\right)}\right] \qquad (2.2)$$

Provided that

(i) $C_j,t_j,U_j(j=1,...,k)$ and σ are real and positive,

(ii) $\sum_{j=1}^{k}\left[\frac{s_j}{t_j}+\frac{U_j}{t_j}.\min_{1\leq j\leq M}\operatorname{Re}\left(\frac{b_j}{\beta_j}\right)\right]>0$ and

$$\operatorname{Re}(\rho)+\sigma.\min_{1\leq j\leq M}\operatorname{Re}\left(\frac{b_j}{\beta_j}\right)>0$$

(iii) $\Omega>0,\left|\arg z\right|<\frac{1}{2}\pi\,\Omega$

where

$$\Omega=\sum_{J=1}^{M}\beta_j+\sum_{j=1}^{N}A_j\alpha_j-\sum_{j=M+1}^{Q}B_j\beta_j-\sum_{j=N+1}^{P}\alpha_j>0 \qquad (2.3)$$

Obviously $f(x_1,...,x_k)$ is not a non-negative function for all positive values of parameters but there exist a number of sets of parameters for which

$$f(x_1,...,x_k)>0,x_j\geq 0\left(j=1,...,k\right),\sum_{j=1}^{k}C_jx_j^{t_j}\leq 1\ \ and\int\underset{R}{\cdots}\int f(x_1,...,x_k)dx_1...dx_k=0.$$

Hence $f(x_1,...,x_k)$ in (2.1) in restricted to those parameter values only.

It is not out of place of mentioned here that the probability density function considered here contains (as particular case) a large variety of elementary function introduced in the literature from time to time. Thus over findings unify and extend the classical statistical research workers. Indeed, as long as one finds the practical situations, when introduction of a more general function is justifiable, the generalization can be put to practical use.

If $f(x_1,...,x_k)$ is a probability density function, then it should satisfy the relation.

$$\int \underset{R}{\dots} \int f(x_1,...,x_k)dx_1...dx_k = 1 \quad (2.4)$$

Putting the values of $f(x_1,...,x_k)$ from (2.1) in (2.4) and evaluating the resulting integral with the help of the following result:

$$\int \underset{R}{\dots} \int \prod_{j=1}^{k}\left\{x_j^{s_j-1}\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\lambda}\right\}\overline{H}\left[y\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\sigma}\right]dx_1...dx_k =$$

$$\prod_{j=1}^{k}\left\{\frac{(C_j)^{s_j/t_j}}{t_j}\right\}\overline{H}^{M,N+k+1}_{P+k+1,Q+1}\left[y\prod_{j=1}^{k}(C_j)^{-U_j/t_j}\left|\begin{matrix}\left(1-\frac{s_j}{t_j},\frac{U_j}{t_j};1\right)_{1,k},(-\lambda,\sigma;1),A^* \\ B^*,\left(-\lambda-\sum_{j=1}^{k}\frac{s_j}{t_j},\sum_{j=1}^{k}\left(\frac{U_j}{t_j}\right)+\sigma;1\right)\end{matrix}\right.\right] \quad (2.5)$$

We easily arrive at the desired value of k^{-1} given by (2.2).

The integral (2.5) which holds true under the conditions mentioned with (2.2).In (2.1) replacing M,N,P,Q respectively by 1,P,P,Q+1, the $\overline{H}$-function reduces to the Wright's generalized hyper geometric function ${}_P\overline{\psi}_Q$ ([6],p.271,(7)), and more if we take $C_j = t_j = U_j = 1 (j = 1,...,k), A_j = B_j = 1$ in (2.2) and use a known result [(Srivastava *et.al.*,1982)p.18,(2.6.3)] therein and then let $a \to 0$, we get probability density function considered by Extons [(1978),p.222,(7.2.1.1)]. For $A_j = B_j = 1$, we get probability density function considered by Goyal and Audich Sunil [(1991), p.78].

The mathematical expectation

Here we shall find Mathematical expectation of multivariate function involving a general class of polynomials. Suppose that

$$g(x_1,...,x_k) = \left(1-\sum_{j=1}^{k}D_J x_j^{t_j}\right)^{\mu} S_n^m\left[y\left(1-\sum_{j=1}^{k}D_j x_j^{t_j}\right)^{\delta}\prod_{j=1}^{k}(x_j)^{\lambda_j}\right] \quad (3.1)$$

Now the Mathematical expectation of $g(x_1,...,x_k)$ for the generalized Dirichlet distribution (2.2) is given by :

$$< g(x_1,...,x_k) >= K\int \underset{R}{\dots} \int \prod_{j=1}^{k}(x_j)^{s_j-1}\left(1-\sum_{j=1}^{k}D_j x_j^{t_j}\right)^{\mu}\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\rho-1} \quad (3.2)$$
$$S_n^m\left[y\left(1-\sum_{j=1}^{k}D_j x_j^{t_j}\right)^{\delta}\prod_{j=1}^{k}(x_j)^{\lambda_j}\right]\overline{H}\left[a\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\sigma}\prod_{j=1}^{k}x_j^{U_j}\right]dx_1...dx_k$$

To evaluated the above integral (3.2), use series representation (1.3) for $S_n^m[x]$ and change the order of integrations and summations and a known result [(Srivastava, 1972), p.18,(2.6.4)] therein , we find that

$$< g(x_1,...,x_k) >= K\sum_{S=0}^{[n/m]}\frac{(-n)_{ms}}{s!}A_{n,s}y^s\int \dots_R \int \prod_{j=1}^{k}x_j^{s_j+\lambda_j s-1} \quad (3.3)$$
$$\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\rho-1}{}_1F_0\left(-\mu-\delta s;-;\sum_{j=1}^{k}D_j x_j^{t_j}\right)\overline{H}\left[a\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\sigma}\prod_{j=1}^{k}x_j^{t_j}\right]dx_1...dx_k$$

Now using the known result [(Srivastava & Karlsson, 1985), p.39, (30) and p.38 (2.4)], changing the order of integration and summation therein and evaluating the resulting multiple integral with the help of (2.5), we finally arrive at the following result:

$$< g(x_1,...,x_k) >= K\sum_{M_1,...,M_{k=0}}^{\infty}\sum_{s=0}^{\left[\frac{n}{m}\right]}\frac{(-n)_{ms}A_{n,s}(-\mu-\delta s)_{M_1+...+M_k}}{s!M_1!...M_k!}y^s \quad (3.4)$$
$$\prod_{j=1}^{k}\left\{\frac{(C_j)^{-(s_j+\lambda_j s)/t_j}}{t_j}\left(\frac{D_j}{C_j}\right)^{M_j}\right\}\overline{H}\left[a\prod_{j=1}^{k}C_j^{-U_j/t_j}\left|\begin{matrix}\left(1-\frac{s_j+\lambda_j s+t_j M_j}{t_j},\frac{U_j}{t_j};1\right)_{1,k},(1-\rho,\sigma;1),A^* \\ B^*,\left(1-\rho-\sum_{j=1}^{k}\left(\frac{s_j+\lambda_j s+\mu_j t_j}{t_j}\right),\sum_{j=1}^{k}\left(\frac{U_j}{t_j}\right)+\sigma;1\right)\end{matrix}\right.\right]$$

Where k is given by (2.2) and providing that

$$(i) \qquad \sum_{j=1}^{k}\left[\frac{s_j}{t_j}+\frac{U_j}{t_j}\min_{1\le j\le M}\mathrm{Re}\left\{b_j/\beta_j+\left(\lambda_j/t_j\right)s\right\}\right]>0,$$

$$\mathrm{Re}(\rho)+\min_{1\le j\le M}\left\{\mathrm{Re}\left(b_j/\beta_j\right)\right\}>0, \qquad (s=0,1,...,n/m);$$

$$(ii) \qquad \delta>0, \mathrm{Re}(\mu)>-1, |D_1|+...+|D_k|<1;$$

(*iii*) The sets (i) and (iii) of conditions given just below (2.2) are satisfied;

(*iv*) $\max\limits_{1\le j\le k}\left\{\left|D_j/C_j\right|\right\}<1$

The result due to Exton [(1973), p.223, (7.2.16)], and Goyal and Audich Sunil [(1991), p.80] can be deduced as a special case of our result (3.4).

The distribution function

The distribution function $F(x_1,...,x_k)$ as the cumulative probability function for the probability density function $F(x_1,...,x_k)$ is given by

$$F(u_1,...,u_k) = k\int_0^{u_1}...\int_0^{u_k}\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\rho-1}\prod_{j=1}^{k}\left(x_j\right)^{s_j-1}\overline{H}\left[a\left(1-\sum_{j=1}^{k}C_j x_j^{U_j}\right)^{\sigma}\prod_{j=1}^{k}x_j^{U_j}\right]dx_1...dx_k \quad (4.1)$$

To Evaluate the integral involved in (4.1), we write the $\overline{H}$-function in terms of Mellin-Barnes integral, change the order of integration and use a known result [(Srivastava *et al.*, 1982),p.13,(2.6.4)], therein, we find that

$$F(u_1,...,u_k) = \frac{K}{2\pi i}\int_{i\infty}^{i\infty}\overline{\Phi}(\xi)a^{\xi}\left\{\int_0^{u_1}...\int_0^{u_k}\prod_{j=1}^{k}x_j^{s_j+U_j\xi-1}\left(1-\sum_{j=1}^{k}C_j x_j^{t_j}\right)^{\rho+a\xi-1}dx_1...dx_k\right\}d\xi \quad (4.2)$$

Where now using the result [(Srivastava & Karlsson, 1985), p.39, (24) in (4.2)] change the order of integrations and summation, evaluating the $x_1,...,x_k$ integrals separately and expressing the resulting contour integral in terms of the $\overline{H}$-Function, we finally obtain:

$$F(u_1,...,u_k) = K \sum_{M_1,...,M_k=0}^{\infty} \prod_{j=1}^{k} \left\{ \frac{U_j^{s_j+M_j t_j} C_j^{M_j}}{M_j\,!} \right\} \overline{H}_{P+k+1,Q+k+1}^{M+1,N+k+1} \left[a \prod_{j=1}^{k} u_j^{U_j} \left| \begin{matrix} (1-s_j-M_j t_j, U_j:1)_{1,k} (1-p,\sigma;1), A^* \\ (1-p+M_1+...+M_k,\sigma), B^*, (-s_j-M_j t_j, U_j;1)_{1,k} \end{matrix} \right. \right] \quad (4.3)$$

Where k is given by (2.2) and the result (4.3) hold true under the following (sufficient) conditions.

(i) The set of conditions (i) and (iii) mentioned just below of (2.2) are satisfied.

(ii) $\mathrm{Re}(p) > 0, \mathrm{Re}\left(s_j + U_{j 1\le j\le M}^{\min} \left| Re\left(b_j \big/ s_j \right) \right. \right) > 0, (j = i,...,k)$

(iii) $|C_1| + ... + |C_k| < 1,$

(iv) $\min\limits_{1\le j\le k} \left\{ \left| U_j^{t_j} C_j \right| \right\} < 1.$

The characteristic function

Suppose that $(x_1,...,x_k)$ is a k-dimensional random variable with density function $f(x_1,...,x_k)$ given by (2.2).Then the characteristic function is given by Exton [(1978), P.232, (7.4.3.1)]:

$$\phi(u_1,...,u_k) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp\left[i(u_1 x_1 + ... + u_k x_k) \right] f(x_1,...x_k) dx_1 ... dx_k \quad (5.1)$$

$$\phi(u_1,...u_k) = K \int_R ... \int e^{i(u_1 x_1 + ... + u_k x_k)} \left(1 - \sum_{j=1}^{k} C_j x_j^{t_j} \right)^{\sigma - 1}$$

$$\prod_{j=1}^{k} x_j^{s_j - 1} \overline{H} \left[a \left(1 - \sum_{j=1}^{k} C_j x_j^{t_j} \right)^{\sigma} \prod_{j=1}^{k} x_j^{U_j} \right] dx_1 ... dx_k =$$

$$K \sum_{M_1,...,M_{k=0}}^{k} \prod_{j=1}^{k} \frac{u_j^{M_j} (C_j)^{-(s_j+M_j)/t_j}}{t_j\,! M_j\,!} \overline{H}_{P+k+1,Q+1}^{M,N+k+1} \left[a \prod_{j=1}^{k} C_j^{-U_j/t_j} \left| \begin{matrix} \left(1 - \frac{s_j+M_j}{t_j}, \frac{U_j}{t_j};1 \right), (1-\rho,\sigma;1), A^* \\ B^*, \left(1 - \rho - \sum_{j=1}^{k} \frac{s_j+M_j}{t_j}, \sum_{j=1}^{k} \frac{U_j}{t_j} + \sigma;1 \right) \end{matrix} \right. \right] \quad (5.2)$$

Provided that the conditions mentioned just below (2.2) are satisfied. Also K is defined by (2.2) and multiply series involved in (5.2) converges absolutely for all values of u_j and C_j.

Lastly, we remark in passing that the distribution function and the characteristic function obtained by Srivastave *et.al.*, [(1982), p.81 and 82] and Goyal & Audich Sunil [(1991), p.224 and 232] can be deduced as particular cases of our functions (4.3) and (5.2) respectively.

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