Indian Journal of Science and Technology, *Vol 10(37), DOI: 10.17485/ijst/2017/v10i37/93435, October 2017*
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Numerical Integration of Analytic Function Through Extrapolation

Arjun Singh*, Saumya Ranjan Jena and Bhupati B. Mishra

Department of Mathematics, School of Applied Sciences, KIIT University, Bhubaneswar – 751024, Odisha, India; arjunatbbsr@gmail.com, saumyafma@kiit.ac.in, bbmishrafma@kiit.ac.in

Abstract

In recent years mixed quadrature rule for analytic functions has been utilized in the field of Science and Technology. A mixed quadrature rule for degree of precision eleven has been established by taking two constituent rules each of degree of precision nine. The proposed mixed quadrature rule has been tested on different functions and comparable with different authors to find the best approximate solution to the exact results. Mixed quadrature rule on analytic function is applied in the field of adaptive quadrature which is essential in the physical sciences.

Keywords: Analytic Functions, Degree of Precision, Maclaurin's Theorem, Mixed Quadrature, Richardson Extrapolation
MSC 2010: 65D30, 65D32

1. Introduction

Here we have mixed Birkhoff-Young modified rule which is obtained using Richardson extrapolation and Gauss-Legendre-4 point transformed quadrature rule each of precision seven to obtain a rule of precision nine keeping the fact in the mind[1-5]. This rule has been convexed with another mixed quadrature rule i.e. (Bull's rule, Birkhoff-Young rule and Gauss-Legendre-4 point rule) to form a mixed quadrature rule $R_{RM\frac{BY}{2}GL4BLBYGL4}(f)$ for numerical integration of type

$$I(f)=\int_L f(z)\,dz \tag{1}$$

Where L is a directed line segment from the point (z_0-h) to (z_0+h) in the complex plane and $f(z)$ is analytic in certain domain Ω containing the line segment L.Lether L5using the transformation $z=(z_0+th)$, where $t\in[-1,1]$ transform the integral (1) to the integral

$$h\int_{-1}^{1} f(z_0+th)\,dt \tag{2}$$

This paper is organized as follows. The mixed quadrature rule of modified Birkhoff -Young rule by Richardson extrapolation with Gauss-Legendre-4 point transformed rule $R_{RM\frac{BY}{2}GL4}(f)$ and its error $E_{RM\frac{BY}{2}GL4}(f)$ has been placed in Section-2. In Section-3 mixed quadrature rule $R_{BLBYGL4}(f)$ has been constructed. The convex combination of $R_{RM\frac{BY}{2}GL4}(f)$ and $R_{BLBYGL4}(f)$ have been made to construct a mixed quadrature rule $R_{RM\frac{BY}{2}GL4BLBYGL4}(f)$ in section-4. The numerical verification of proposed rule and a comparison with numerical results of different authors has been focused in section - 5. Section - 6 follows the conclusion.

2. Constructioon of Modified Birkhoff-Young Rule through Richardson Extrapolation

The Birkhoff-Young rule for analytic functions.

*Author for correspondence

$$R_{BY}(f)=\frac{h}{15}\left[24f(z_0)+4\{f(z_0+h)+f(z_0-h)\}-\{f(z_0+ih)+f(z_0-ih)\}\right] \tag{3}$$

Applying Maclaurin's series in Equation (3)

$$R_{BY}(f)=2h\begin{bmatrix} f(z_0)+\frac{h^2}{3!}f^{ii}(z_0)+\frac{h^4}{5!}f^{iv}(z_0)+\frac{h^6}{3\times 6!}f^{vi}(z_0) \\ +\frac{h^8}{5\times 8!}f^{viii}(z_0)+\frac{h^{10}}{3\times 10!}f^{x}(z_0)+\frac{h^{12}}{5\times 12!}f^{xii}(z_0)+\dots\dots \end{bmatrix} \tag{4}$$

2.1 Error in Birkhoff-Young Rule

$(E_{BY}(f))$

$$I(f)=R_{BY}(f)+E_{BY}(f) \tag{5}$$

Using Equation (1) and Equation (4) in Equation (5)

$$E_{BY}(f)=-\frac{8h^7}{21\times 6!}f^{vi}(z_0)-\frac{8h^9}{45\times 8!}f^{viii}(z_0)-\frac{16h^{11}}{33\times 10!}f^{x}(z_0)-\frac{16h^{13}}{65\times 12!}f^{xii}(z_0)\dots\dots \tag{6}$$

2.2 Richardson Extrapolation for Modified Birkhoff-Young Rule

$\left(R_{RM\frac{BY}{2}}(f)\right)$

For n number of sub-intervals

$$E_{RBY}(f)=I(f)-I_n(f) \tag{7}$$

Where, $I_n(f)=R_{BY}(f)$

$$E_{R\frac{BY}{2}}(f)=I(f)-I_{\frac{n}{2}}(f) \tag{8}$$

$I_{\frac{n}{2}}(f)=R_{\frac{BY}{2}}(f)$ From Equation (6),

$$E_{RBY}(f)=-\frac{h^7}{1890}f^{vi}(z_0)-\frac{h^9}{226800}f^{viii}(z_0)-\frac{h^{11}}{7484400}f^{x}(z_0)-\frac{h^{13}}{1945944000}f^{xii}(z_0)\dots\dots \tag{9}$$

$$E_{R\frac{BY}{2}}(f)=-\frac{2^7h^7}{1890}f^{vi}(z_0)-\frac{2^9h^9}{226800}f^{viii}(z_0)-\frac{2^{11}h^{11}}{7484400}f^{x}(z_0)-\frac{2^{13}h^{13}}{1945944000}f^{xii}(z_0)\dots\dots \tag{10}$$

Now multiplying (2^7) with Equation (7) and subtracting it from Equation (8).

$$\left\{I(f)-I_{\frac{n}{2}}(f)\right\}-2^7\{I(f)-I_n(f)\}=E_{R\frac{BY}{2}}(f)-2^7E_{RBY}(f) \tag{11}$$

Using Equation (9) and Equation (10) in Equation (11).

$$I(f)=\left[\frac{128I_n(f)-I_{\frac{n}{2}}(f)}{127}\right]+\left[\begin{array}{l}\frac{2^7h^9}{127\times5\times15120}f^{viii}(z_0)+\frac{2^7h^{11}}{33\times127\times15120}f^{x}(z_0)+\\ \frac{2^7h^{13}}{125\times16\times127\times108\times143}f^{xii}(z_0).......\end{array}\right] \tag{12}$$

$$I(f)=R_{RM\frac{BY}{2}}(f)+E_{RM\frac{BY}{2}}(f) \tag{13}$$

$$R_{RM\frac{BY}{2}}(f)=\left[\frac{128I_n(f)-I_{\frac{n}{2}}(f)}{127}\right] \tag{14}$$

$$E_{RM\frac{BY}{2}}(f)=\left[\frac{2^7h^9}{127\times5\times15120}f^{viii}(z_0)+\frac{2^7h^{11}}{33\times127\times15120}f^{x}(z_0)+\frac{2^7h^{13}}{125\times16\times127\times108\times143}f^{xii}(z_0).......\right] \tag{15}$$

Where Equation (14) and Equation (15) are called Modified Birkhoff-Young rule due to Richardson extrapolation and Error in Modified Birkhoff-Young rule respectively.

2.3 Gauss-Legendre-4 Point Transformed Rule

The Gauss-Legendre-4 point transformed rule is,

$$R_{GL4}(f)=\frac{h}{36}\left[\left(18+\sqrt{30}\right)\left\{f(z_0-\alpha h)+f(z_0+\alpha h)\right\}+\left(18-\sqrt{30}\right)\left\{f(z_0-\beta h)+f(z_0+\beta h)\right\}\right] \tag{16}$$

$$\alpha=\sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}\quad,\quad\beta=\sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$$

Applying Maclaurin's expansion in Equation (16)

$$R_{GL4}(f)=2h\left[\begin{array}{l}f(z_0)+\frac{h^2}{3!}f^{ii}(z_0)+\frac{h^4}{5!}f^{iv}(z_0)+\frac{h^6}{7!}f^{vi}(z_0)+\frac{h^8}{8!}f^{viii}(z_0)\left(\frac{6321}{7^4\times5^2}\right).\\ +\frac{h^{10}}{10!}f^{x}(z_0)\left(\frac{32781}{7^5\times5^2}\right)+\frac{h^{12}}{12!}f^{xii}(z_0)\left(\frac{850689}{7^6\times5^3}\right)........\end{array}\right] \tag{17}$$

2.4 Error in Gauss-Legendre-4 Point Transformed Rule

$\left(E_{GL4}(f)\right)$

$$I(f)=R_{GL4}(f)+E_{GL4}(f) \tag{18}$$

Using Equation (1) and Equation (17)

$$E_{GL4}(f)=\left(\frac{6272}{540225\times8!}\right)h^9f^{viii}(z_0)+\left(\frac{119168}{4621925\times10!}\right)h^{11}f^{x}(z_0)+\left(\frac{7294336}{191179625\times12!}\right)h^{13}f^{xii}(z_0)....... \tag{19}$$

3. Mixed Quadrature Rule

$$\left(R_{RM\frac{BY}{2}GL4}(f)\right)$$

Mixed quadrature rule for modified Birkhoff-Young rule using Richardson extrapolation and Gauss-Legendre-4 point transformed rule has been constructed.

Multiplying Equation (13) and Equation (18) by $\left(\frac{1}{5880}\right)$ and $\left(-\frac{1}{127}\right)$ respectively and adding them

$$I(f)=\frac{1}{5753}\left(5880R_{GL4}(f)-127R_{RM\frac{BY}{2}}(f)\right)+\frac{1}{5753}\left(5880E_{GL4}(f)-127E_{RM\frac{BY}{2}}(f)\right)$$

$$R_{RM\frac{BY}{2}GL4}(f)=\frac{1}{5753}\left(5880\,R_{GL4}(f)-127R_{RM\frac{BY}{2}}(f)\right) \quad (21)$$

$$E_{RM\frac{BY}{2}GL4}(f)=\frac{1}{5753}\left(5880\,E_{GL4}(f)-127E_{RM\frac{BY}{2}}(f)\right) \quad (22)$$

Where Equation (21) and Equation (22) are called mixed quadrature of modified Birkhoff-Young and Gauss-Legendre-4 point rule due to Richardson extrapolation and error due to mixed quadrature rule respectively.

Using Equation (15) and Equation (19) in Equation (22)

$$E_{RM\frac{BY}{2}GL4}(f)=-\left(\frac{253152}{135\times5753\times11\times10!}\right)h^{11}f^{x}(z_0)-\left(\frac{481134551}{49\times3\times143\times5753\times13\times12!}\right)h^{13}f^{xii}(z_0)\ldots. \quad (23)$$

3.1 Mixed Quadrature Rule of Degree of Precision of Nine

The Boole's rule $\left(R_{BL}(f)\right)$

$$I(f)\cong R_{BL}(f)=\frac{h}{45}\left[7\left\{f(z_0-h)+f(z_0+h)\right\}+32\left\{f\left(z_0-\frac{h}{2}\right)+f\left(z_0+\frac{h}{2}\right)\right\}+12f(z_0)\right] \quad (24)$$

Expanding each term of Equation (24) by Maclaurin's expansion

$$R_{BL}(f)=2hf(z_0)+\frac{2h^3}{3!}f^{ii}(z_0)+\frac{2h^5}{5!}f^{iv}(z_0)+\left(\frac{1}{3\times6!}\right)h^7f^{vi}(z_0)$$

$$+\left(\frac{57}{4\times45\times8!}\right)h^9f^{viii}(z_0)+\left(\frac{5}{16\times10!}\right)h^{11}f^{x}(z_0)+\left(\frac{897}{64\times45\times12!}\right)h^{13}f^{xii}(z_0) \quad (25)$$

$$I(f)=R_{BL}(f)+E_{BL}(f) \quad (26)$$

Multiplying (-8) with Equation (26) and adding it with Equation (5)

$$I(f)=\frac{1}{7}\left[8R_{BL}(f)-R_{BY}(f)\right]+\frac{1}{7}\left[8E_{BL}(f)-E_{BY}(f)\right] \quad (27)$$

$$R_{BLBY}(f)=\frac{1}{7}\left[8R_{BL}(f)-R_{BY}(f)\right] \tag{28}$$

$$E_{BLBY}(f)=\frac{1}{7}\left[8E_{BL}(f)-E_{BY}(f)\right] \tag{29}$$

Where Equation (28) and Equation (29) are called mixed quadrature of Boole's and Birkhoff rule and Error due to mixed quadrature rule respectively.

From Equation (29)

$$E_{BLBY}(f)=-\left(\frac{26}{315\times 8!}\right)h^9 f^{viii}(z_0)-\left(\frac{37}{462\times 10!}\right)h^{11} f^{x}(z_0)-\left(\frac{4749}{32760\times 12!}\right)h^{13} f^{xii}(z_0)\ldots\ldots \tag{30}$$

$$I(f)=R_{BLBY}(f)+E_{BLBY}(f) \tag{31}$$

Multiplying $\left(\frac{64}{35}\right)$ in Equation (31) and (13) in Equation (18) respectively and adding them

$$I(f)=\frac{1}{519}\left[455R_{GL4}(f)+64R_{BLBY}(f)\right]+\frac{1}{519}\left[455E_{GL4}(f)+64E_{BLBY}(f)\right] \tag{32}$$

$$I(f)=R_{BLBYGL4}(f)+E_{BLBYGL4}(f) \tag{33}$$

$$R_{BLBYGL4}(f)=\frac{1}{519}\left[455R_{GL4}(f)+64R_{BLBY}(f)\right] \tag{34}$$

$$E_{BLBYGL4}(f)=\frac{1}{519}\left[455E_{GL4}(f)+64E_{BLBY}(f)\right] \tag{35}$$

Where Equation (34) and Equation (35) are called mixed quadrature rule due to $\left(R_{GL4}(f)\right)$ and $\left(R_{BLBY}(f)\right)$ and Error $\left(E_{BLBYGL4}(f)\right)$ due to them.

4. A Mixed Quadrature Due to Richardson Extrapolation

Here a mixed quadrature rule for degree of precision eleven has been constructed

Multiplying $\left(\frac{1669}{25431}\right)$ and $\left(\frac{879}{5753}\right)$ with Equation (20) and Equation (33) respectively and adding them

$$I(f)=\frac{1}{31955606}\left(9601757R_{RM\frac{BY}{2}GL4}(f)+22353849R_{BLBYGL4}(f)\right)$$
$$+\frac{1}{31955606}\left(9601757E_{RM\frac{BY}{2}GL4}(f)+22353849E_{BLBYGL4}(f)\right) \tag{36}$$

$$I(f)=R_{RM\frac{BY}{2}GL4BLBYGL4}(f)+E_{RM\frac{BY}{2}GL4BLBYGL4}(f) \tag{37}$$

$$R_{RM\frac{BY}{2}GL4BLBYGL4}(f)=\frac{1}{31955606}\left(9601757\ R_{RM\frac{BY}{2}GL4}(f)+22353849\ R_{BLBYGL4}(f)\right) \tag{38}$$

$$E_{RM\frac{BY}{2}GL4BLBYGL4}(f)=\frac{1}{31955606}\left(9601757\ E_{RM\frac{BY}{2}GL4}(f)+22353849\ E_{BLBYGL4}(f)\right) \tag{39}$$

Where Equation (38) and Equation (39) are called mixed quadrature due to Richardson extrapolation and error due to them respectively.

$$\left|E_{RM\frac{BY}{2}GL4BLBYGL4}(f)\right|=\frac{77347081}{954174423\times 12!}h^{13}\left|f^{xii}(z_0)\right| \tag{40}$$

Degree of precision of the above mixed quadrature rule is eleven.

Theorem

The error bound for the truncation error arises due to mixed quadrature rule has been constructed.

$$E_{RM\frac{BY}{2}GL4BLBYGL4}(f)=I(f)-R_{RM\frac{BY}{2}GL4BLBYGL4}(f)$$

is given by $\left|E_{RM\frac{BY}{2}GL4BLBYGL4}(f)\right|\le\frac{845021376\,h^{11}M}{217262246355\times 10!}\left|\eta_2-\eta_1\right|,\qquad \eta_1,\eta_2\in[-1,1]$

Where, $M=\underset{-1\le x\le 1}{Max}\left|f^{xi}(z_0)\right|$

Proof:

$$E_{RM\frac{BY}{2}GL4}(f)\cong\frac{32\times 1669\times 879\times 9}{49\times 15\times 11\times 519\times 5753\times 10!}h^{11}f^{x}(\eta_1),\qquad \eta_1\in[-1,1]$$

$$E_{RM\frac{BY}{2}GL4}(f)\cong\frac{253152\times 1669}{135\times 5753\times 11\times 25431\times 10!}h^{11}f^{x}(\eta_2),\qquad \eta_2\in[-1,1]$$

$$E_{RM\frac{BY}{2}GL4BLBYGL4}(f)\cong\frac{422510688}{217262246355\times 10!}h^{11}\left[f^{x}(\eta_2)-f^{x}(\eta_1)\right]$$

where $K=\max_{-1\le x\le 1}\left|f^{x}(z_0)\right|$ and $k=\min_{-1\le x\le 1}\left|f^{x}(z_0)\right|$. As $f^{x}(z_0)$ is continuous &$[-1,\ 1]$ is compact. Hence there exists points b and a in the interval $[-1,\ 1]$ such that $K=f^{x}(b)$ and $k=f^{x}(a)$. Thus by[6]

$$\left|E_{RM\frac{BY}{2}GL4BLBYGL4}(f)\right|=\left|\frac{422510688}{217262246355\times 10!}h^{11}\int_{\eta_1}^{\eta_2}f^{xi}(z_0)\,dz\right|\le\frac{422510688}{217262246355\times 10!}\int_{\eta_1}^{\eta_2}\left|f^{xi}(z_0)\right|dz$$

By mean value theorem $|b-a|\le 2$

$$\left|E_{RM\frac{BY}{2}GL4BLBYGL4}(f)\right|\le\frac{845021376\,h^{11}M}{217262246355\times 10!}\left|\eta_2-\eta_1\right|,\qquad \eta_1,\eta_2\in[-1,1]$$

Which gives only a theoretical error bound as η_1 and η_2 are known points in $[-1,1]$.

The Equation (1) has proved that the error will be less if the points η_1 and η_2 are closer to each other.

5. Numerical Verification

The numerical examples has been done taking suitable examples.

$$I_1 = \int_{-i}^{i} e^{z}\ dz, \qquad I_2 = \int_{-i}^{i} \cos z\ dz,\ \ I_3 = \int_{\frac{-i}{3}}^{\frac{i}{3}} \cosh z\ dz$$

Table 1 shows Numerical results of different functions are compared with their exact values for different rules.
It is evident that,

$$\left|E_{RM\frac{BY}{2}GL4BLBYGL4}(f)\right| '' \ \left|E_{BLBYGL4}(f)\right| '' \ \left|E_{RM\frac{BY}{2}GL4}(f)\right| '' \ \left|E_{RM\frac{BY}{2}}(f)\right|$$

Table 1. (Numerical results of different functions are compared with their exact values for different rules)

Quadrature rule	Approximation value I_1	Approximation value I_2	Approximation value I_3
$R_{BY}(f)$	1.682417145154309 i	2.350936031119045 i	0.654389151885734 i
$R_{RM\frac{BY}{2}}(f)$	1.682930690528399 i	2.350387003041248 i	0.654389391594494 i
$R_{BL}(f)$	1.682878138736396 i	2.350470903567373 i	0.654389363469878 i
$R_{BLBY}(f)$	1.682943994962409 i	2.350404456776563 i	0.654389393698184 i
$R_{GL4}(f)$	1.682941688695974 i	2.350402092156376 i	0.654389393577715 i
$R_{RM\frac{BY}{2}GL4}(f)$	1.682941931485350 i	2.350402425255215 i	0.654389393621495 i
$R_{BLBYGL4}(f)$	1.682941973091064 i	2.350402383747305 i	0.654389393592324 i
$R_{RM\frac{BY}{2}GL4BLBYGL4}(f)$	1.682941960589722 i	2.350402396219260 i	0.654389393601089 i
Exact Value	1.682941969615793 i	2.350402387287603 i	0.654389393592304 i
$\left\|E_{RM\frac{BY}{2}GL4BLBYGL4}(f)\right\|$	0.000000009026071 i	0.000000008931657 i	0.000000000008785 i

The numerical result of I_1 for different researchers, $R_{DJ}(f)$[7], $R_{JD}(f)$[8], $R_{MD}(f)$[9], $R_{PD}(f)$[10], $R_{SAA}(f)$[11] the respective approximate value of I_1 has been in Table 2 numerically calculated.

$$I_1 = \int_{-i}^{i} e^z \ dz$$

Table 2. (Comparative study of I_1 with the proposed rule with different researchers)

Quadrature rule	Approximation Value
$R_{DJ}(f)$	1.6829423i
$R_{JD}(f)$	1.682941652305530i
$R_{MD}(f)$	1.682944i
$R_{SAA}(f)$	1.6829419140i
Exact Value	1.682941969615793i

6. Conclusion

From Numerical result it is evident that the present mixed quadrature rule $R_{RM\frac{BY}{2}GL4BLBYGL4}(f)$ of degree of precision eleven is more efficient and applicable than all the mixed quadrature rules by different authors. The proposed mixed quadrature rule numerically integrates more accurately I_1 to the exact result.

7. References

1. Acharya BP, Dash RN. Compound Birkhoff-Young rule for numerical integration of analytic functions. International Journal of Mathematical Education in Science and Technology. 1983; 14(1):1-10. Crossref.
2. Birkhoff G, Young D. Numerical quadrature of analytic and harmonic functions. Journal of Mathematical Physics. 1950; 29(1-4):217. Crossref.
3. Lether FG. On Birkhoff-Young quadrature of analytic functions. Journal of Computational and Applied Mathematics. 1976; 2(2):81-4. Crossref.
4. Uma K, Murugusundaramoorthy G, Vijaya K. Analytic Functions of Complex Order Defined by Fractional Integrals Involving Foxs H-Functions. Indian Journal of Science and Technology. 2015 Jul; 8(16):1-7.
5. Alipour R. A New Numerical Method for Optimal Control of a Class of Hybrid Systems. Indian Journal of Science and Technology. 2014 Jan; 7(9):1-7.
6. Conte S. Boor Cde. Tata Mc-Graw Hill: Elementary Numerical Analysis. 1980; p. 1-18. PMid:6892821.
7. Dash RB, Jena SR. Mixed quadrature of Birkhoff-Young and Weddle's transformed rule. Indian Journal of Mathematics and Mathematical Sciences. 2009; 5(1):29-32.
8. Jena SR, Dash P. Numerical treatment of analytic functions via mixed quadrature rule. Research Journal of Applied Sciences, Engineering and Technology. 2015; 10(4):391-2. Crossref.
9. Mohanty S, Dash RB. A mixed quadrature rule for numerical integration of analytic functions by using Birkhoff-Young and Bools quadrature. International Journal of Mathematical Science Education. 2010; 3(1):31-5.
10. Pati A, Dash RB. A mixed quadrature rule for numerical integration of analytic functions. Journal of Computer and Mathematical Sciences. 2012; 3(5):526-9.
11. Senapati S, Acharya M, Acharya BP. On Numerical quadrature of analytic functions. Applied Mathematical Science. 2011; 5(69):3411-20.