Successive Separation Procedure of Reduction for Nonlinear Large Scale Systems

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Abstract

For controller implementation and due to practical considerations a reduced order model is usually needed. So in this work, the problem of order reduction of nonlinear multi time scales systems in the framework of singular perturbations is addressed. A successive separation algebraic procedure is proposed to derive a polynomial reduced model by using the Kronecker product based state representation and the properties of the tensor algebra.

Keywords: Kronecker Product, Polynomial Reduced Order Model, Nonlinear Large-scale Systems, Singular Perturbation Theory.

1. Introduction

Due to not only theoretical interest, but also to the relevance of this topic to control engineering applications, the singular perturbations techniques have received a great deal of attention over the past two decades[1-7]. Several results are developed in the literature to cope with order reduction of linear systems and even for particular classes of nonlinear systems[2] using the singular perturbations theory. But no simple and easy implemented approach for order reduction of multi-time-scale interconnected nonlinear systems are provided.

Indeed, the complexity encountered in solving nonlinear differential equations[16] and the lack of rigorous analytical reduced model have motivated the development of the proposed algebraic approach based on the description of the studied systems by an analytical control affine state space equation with polynomial vector fields, based on the use of Kronecker product[9]. The properties of the tensor algebra allow to overcome many difficulties.

The layout of the paper is as follows. In the first section, the theory background of the singular perturbation theory is state briefly. In the next section, the description of the studied systems and the problem statement are formulated. A proposed procedure is also given where the different steps of dynamic separation and construction of the reduced model is detailed. The numerical simulations demonstrate the efficiency of the proposed approach and the accuracy of the approximate reduced model. Finally, some conclusions are drawn.

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2. Materials and Methods

2.1 Singular Perturbation Theory

Let us consider the general nonlinear singularly perturbed systems described by the following state equations:

$$\begin{cases} \dot{x} = F(x,z,t), \quad x(t_0) = x_0, x \in \mathfrak{R}^{n_1} \\ \varepsilon\dot{z} = G(x,z,t), \quad z(t_0) = z_0, z \in \mathfrak{R}^{n_2} \end{cases} \quad (1)$$

where the state variables are divided into n_1 slow states x and n_2 fast states z and ε is a positive small singular perturbed parameter such that $0< \varepsilon <<1$, which quantifies the speed ratio of the slow versus the fast dynamical phenomena of the process, $t \in \mathfrak{R}^+$.

One of the delicate problems is to explicit not defined at fortiori the parameter ε depending on the adopted representation of the studied system.

Let us introduce the following assumptions[8,11].

Assumption 1

The Jacobian $\dfrac{\partial G(x,z,t)}{\partial z}$ at $\varepsilon = 0$ is non singular for all x, z D, where D is a region of interest for the approximate analysis[8].

Assumption 2

For all x, z D, real parts of the eigenvalues of $\dfrac{\partial G(x,z,t)}{\partial z}$ at $\varepsilon = 0$ are smaller than a fixed negative number:

$$\text{Re}\,\lambda\left\{\left.\dfrac{\partial G}{\partial z}\right|_{\mu=0}\right\} < c < 0$$

When the Assumption 1 is satisfied, the slow reduced model is obtained by formally setting $\varepsilon = 0$ in equation (1) as:

$$\begin{cases} \dot{\bar{x}} = F(\bar{x},\bar{z},t) & (a) \\ 0 = G(\bar{x},\bar{z},t) & (b) \end{cases} \quad (2)$$

Solving for $\bar{z}$, the equation (2-b)

$$\bar{z} = \psi(\bar{x}) \quad (3)$$

and substituting $\bar{z}$ (3) into (2) lead to the following model:

$$\dot{\bar{x}} = F(\bar{x},\psi(\bar{x})\ t) = \overline{F}(\bar{x},t) \ , \quad \bar{x}(0) = x_0 \quad (4)$$

It neglects the fast phenomenon, considering that it has instantaneously reached its equilibrium. The fast reduced model is obtained by introducing the fast time scale $\tau = \dfrac{t - t_0}{\varepsilon}$ and the fast variables $\tilde{x}(\tau)$ and $\widetilde{z}(\tau)$ such that:

$$\begin{cases} x(t) = \bar{x}(t) + \tilde{x}(\tau) \\ z(t) = \bar{z}(t) + \tilde{z}(\tau) \end{cases} \quad (5)$$

Expressing the derivatives of x and z with respect to t and τ, substituting them into (14) and letting $\varepsilon \to 0$, we obtain:

$$\dfrac{d\tilde{x}}{d\tau} = 0; \quad \tilde{x}(0) = x(0) - \bar{x}(0) = 0 \quad (6)$$

$$\dfrac{d\tilde{z}}{d\tau} = G(\bar{x}(t)\ \bar{z}(t) + \tilde{z}(\tau)); \quad \tilde{z}(0) = \bar{z}(0) - \tilde{z}(0) \quad (7)$$

where the fast part $\tilde{z}(\tau)$ is the state of the fast reduced model (7), called also the solution of the boundary layer. Its equilibrium $\tilde{z}(\tau) = 0$ is asymptotically stable if assumption 2 is satisfied. If in addition $\tilde{z}(0)$ is in the region of attraction of this equilibrium, then the following two time scales approximation is valid, [8]:

$$\begin{cases} x(t) = \overline{x}(t) + \mathrm{O}(\varepsilon) \\ z(t) = \overline{z}(t) + \tilde{z}(\tau) + \mathrm{O}(\varepsilon) \end{cases} \tag{8}$$

It is shown here that the reduced model will be determined by solving for $\overline{z}$ the equation (2-b). This appears to be easy in the case of linear systems but seems to be a difficult task in the general case of nonlinear systems. Several results are obtained when considering linear systems and particular class of nonlinear systems and some others in the case of nonlinear systems described by state equations with no direct transmission matrix $L(x, z)$ (9).

$$\begin{cases} \dot{x} = F(x, z) + G(x, z)U \\ \varepsilon \dot{z} = H(x, z) + L(x, z)U \end{cases} \tag{9}$$

So the problem we are addressing in this paper can be stated as follows; given a high order multi-time scale system with nonlinear vector functions $F(.),\ G(.),\ H(.)$ and $L(.)$, how can we exploit the singular perturbation theory and the tensors representation to derive in a relatively simple way a simplified model and provide a computational procedure for this aim.

The following sections will show that good results are obtained. The principal key to our work is the use of a powerful mathematical tool in the description of the nonlinear studied systems: the Kronecker product.

2.2 Description of the Studied Systems and Problem Statement

We will focus on realistic models of q time-scales systems which involve interacting dynamic phenomena of widely different speeds and described by the following state separable form:

$$\begin{cases} \varepsilon_1\ \dot{X}_1 = F_1(X_1, \cdots, X_i, \cdots, X_q)\ +\ G_1(X_1, \cdots, X_i, \cdots, X_q)\ U \\ \vdots \\ \varepsilon_i\ \dot{X}_i = F_i(X_1, \cdots, X_i, \cdots, X_q) +\ G_i(X_1, \cdots, X_i, \cdots, X_q)\ U \\ \vdots \\ \varepsilon_q\ \dot{X}_q = F_q(X_1, \cdots, X_i, \cdots, X_q) +\ G_q(X_1, \cdots, X_i, \cdots, X_q)\ U \end{cases}$$

$$\tag{10}$$

where $X_i \in \mathfrak{R}^{n_i}, i = 1, \dots, q$ are the state vectors, $U \in \mathfrak{R}^m$ is the control vector and $F_i(.), G_i\ (.)\ \big|_{i=1,\cdots,q}$, the vector functions, continuously differentiable, are supposed analytic. Moreover, ε_i denotes small non-negative singular perturbation parameters such that $\varepsilon_k < \varepsilon_i$ for $k > i$, making the state variables to be ordered accordingly to their increasing speed. The system (10) can be studied like a multi-parameter system. ε_1 appears here for the aim of symmetry but it is going to be considered equal to 1.

The simplification of the studied system (10) can be done gradually when posing successively $\varepsilon_q = 0$, $\varepsilon_{q-1} = 0, \dots$.

It will appear a recurrence that makes possible to derive, going from the fastest to the slowest subsystem, the degenerate slow model. In each stage of reduction, we apply an analytical identification approach which is

also introduced in this paper to calculate all the matrices appearing in the description of each reduced order model.

3. Results and Discussions

3.1 Algebraic Approach for Order Reduction

The separation of time scales starts with the following model obtained by rewriting the system (10) in a compact form:

$$\begin{cases} \dot{X}_{s1} = F_{s1}(X_{s1}, X_q) + G_{s1}(X_{s1}, X_q)\,U \\ \varepsilon_q\,\dot{X}_q = H_q(X_{s1}, X_q) + L_q(X_{s1}, X_q)\,U \end{cases} \quad (11)$$

where $X_{s1} = [X_1^T \quad X_2^T \quad \cdots \quad X_{q-1}^T]^T \in \Re^{n-n_q}$ is predominantly slow and X_q contains the n_q fastest state variables. n denotes the order of the full studied system defined by $n = \sum_{i=1}^{q} n_i$. $F_{s1}(.)$, $G_{s1}(.)$, $H_q(.)$ and $L_q(.)$, are expressed from the functions $F_i(.)\big|_{i=1,\cdots,q}$ and $G_i(.)\big|_{i=1,\cdots,q}$,(see previous work[15]).

They admit a generalized Taylor series developments using the Kronecker product and power notations as follows

$$\begin{cases} F_{s1}(X_{s1}, X_q) = \sum_{i>1} \sum_{j}^{i+1} F_{ij,1}(X_{s1}^{[i+1-j]} \otimes X_q^{[j-i]}) \\ G_{s1}(X_{s1}, X_q) = \sum_{i>1} \sum_{j}^{i+1} G_{ij,1}(I_m \otimes X_{s1}^{[i+1-j]} \otimes X_q^{[j-i]}) \\ H_q(X_{s1}, X_q) = \sum_{i>1} \sum_{j}^{i+1} H_{ij,1}(X_{s1}^{[i+1-j]} \otimes X_q^{[j-i]}) \\ L_q(X_{s1}, X_q) = \sum_{i>1} \sum_{j}^{i+1} L_{ij,1}(I_m \otimes X_{s1}^{[i+1-j]} \otimes X_q^{[j-i]}) \end{cases}$$

(12)

where $X_{s1}^{[i]}$ and $X_q^{[i]}$ are the i-th redundant Kronecker power of a n_{s1}^i ($n_{s1} = \sum_{i=1}^{q-1} n_i$) dimensional vector respectively n_q^i dimensional vector. $F_{ij,1}, G_{ij,1}, L_{ij,1}$ and $H_{ij,1}$ are constant matrices with appropriate dimensions.

Based on[15], it is proved that the previous development (12) can be truncated at an arbitrary order r. In practice, the choice of the third order polynomial description (r = 3) is considered enough to model many physical processes[12,13,14]. So, all the analytical functions (12) can be approximated by the following vector polynomials:

$$\begin{cases} F_{s1}(X_{s1}, X_q) = F_{11,1}X_{s1} + F_{12,1}X_q + F_{21,1}X_{s1}^{[2]} + F_{22,1}X_{s1} \otimes X_q \\ \qquad + F_{23,1}X_q^{[2]} + F_{31,1}X_{s1}^{[3]} + F_{32,1}X_{s1}^{[2]} \otimes X_q \\ \qquad + F_{33,1}X_{s1} \otimes X_q^{[2]} + F_{34,1}X_q^{[3]} + O_1(X_{s1}^{[4]}, X_q^{[4]}). \\ G_{s1}(X_{s1}, X_q) = G_{11,1}(I_m \otimes X_{s1}) + G_{12,1}(I_m \otimes X_q) + G_{21,1}(I_m \otimes X_{s1}^{[2]}) \\ \qquad + G_{22,1}(I_m \otimes X_{s1} \otimes X_q) + G_{23,1}(I_m \otimes X_q^{[2]}) + G_{31,1}(I_m \otimes X_{s1}^{[3]}) \\ \qquad + G_{32,1}(I_m \otimes X_{s1}^{[2]} \otimes X_q) + G_{33,1}(I_m \otimes X_{s1} \otimes X_q^{[2]}) \\ \qquad + G_{34,1}(I_m \otimes X_q^{[3]}) + O_4(X_{s1}^{[4]}, X_q^{[4]}). \end{cases}$$

$$\begin{cases} H_q(X_{s1}, X_q) = H_{11,1}X_{s1} + H_{12,1}X_q + H_{21,1}X_{s1}^{[2]} + H_{22,1}X_{s1} \otimes X_q \\ \qquad + H_{23,1}X_q^{[2]} + H_{31,1}X_{s1}^{[3]} + H_{32,1}X_{s1}^{[2]} \otimes X_q \\ \qquad + H_{33,1}X_{s1} \otimes X_q^{[2]} + H_{34,1}X_{s1}^{[3]} + O_2(X_{s1}^{[4]}, X_q^{[4]}). \\ L_q(X_{s1}, X_q) = L_{11,1}(I_m \otimes X_{s1}) + L_{12,1}(I_m \otimes X_q) + L_{21,1}(I_m \otimes X_{s1}^{[2]}) \\ \qquad + L_{22,1}(I_m \otimes X_{s1} \otimes X_q) + L_{23,1}(I_m \otimes X_q^{[2]}) + L_{31,1}(I_m \otimes X_{s1}^{[3]}) \\ \qquad + L_{32,1}(I_m \otimes X_{s1}^{[2]} \otimes X_q) + L_{33,1}(I_m \otimes X_{s1} \otimes X_q^{[2]}) \\ \qquad + L_{34,1}(I_m \otimes X_q^{[3]}) + O_4(X_{s1}^{[4]}, X_q^{[4]}). \end{cases}$$

The terms $O_1(X_{s1}^{[4]}, X_q^{[4]})$, $O_2(X_{s1}^{[4]}, X_q^{[4]})$, $O_3(X_{s1}^{[4]}, X_q^{[4]})$, $O_4(X_{s1}^{[4]}, X_q^{[4]})$ are the fourth order error of the previous developments.

- First step of reduction

Setting $\varepsilon_q = 0$ and assuming that it exists a unique root solution of (13-b) in the interesting region which is considered here as a neighborhoods of the equilibrium taken to be the origin, the first generated subsystem verifying the assumption 1 and the assumption 2 is obtained by solving:

$$\begin{cases} \dot{\bar{X}}_{s1} = F_{s1}(\bar{X}_{s1}, \bar{X}_q) + G_{s1}(\bar{X}_{s1}, \bar{X}_q) U & (a) \\ 0 = H_q(\bar{X}_{s1}, \bar{X}_q) + L_q(\bar{X}_{s1}, \bar{X}_q) U & (b) \end{cases} \quad (13)$$

After the first step of reduction, the first obtained reduced model is considered to be described by a polynomial representation to preserve the same description of the studied system

$$\dot{\bar{X}}_{s1} = \sum_{i=1}^{r} A_{i,1} \bar{X}_{s1}^{[i]} + \sum_{i=1}^{r} B_{i,1}(I_m \otimes \bar{X}_{s1}^{[i]}) U + O(\bar{X}_{s1}^{[4]})$$

$$(14)$$

where $O(\bar{X}_{s1}^{[4]})$ is the fourth order error of (14).

The constant matrices $A_{i,1}$ and $B_{i,1}$ are of appropriate dimensions and are calculated by applying the identification approach presented below and:

To get the property of full rank matrices, we first rewrite the equation (14) in the nun-redundant form as:

$$\dot{\hat{X}}_{s1} = \sum_{i=1}^{r} \hat{A}_{i,1} \hat{\bar{X}}_{s1}^{[i]} + \sum_{i=1}^{r} \hat{B}_{i,1}(I_m \otimes \hat{\bar{X}}_{s1}^{[i]}) U \quad (15)$$

where $\hat{A}_{i,1} = A_{i,1} T_i$ and $\hat{B}_{i,1} = B_{i,1}(I_m \otimes T_i)$ (16)

$\bar{X}_{s1}^{[i]}$(resp. $\hat{\bar{X}}_{s1}^{[i]}$) is defined by (A.6) (resp.(A.7)) and T_i^{+} is the pseudo-inverse of T_i expressed in (A.8), (see Appendix).

- Considering the above notations and applying the *vec* function defined in (A.9) and the corresponding properties (A.10), (A.11), (A.12), the equation (15) is rewritten as:

$$vec(\dot{\hat{X}}_{s1}) = \dot{\hat{X}}_{s1} = vec(\sum_{i=1}^{r} \hat{A}_{i,1} \hat{\bar{X}}_{s1}^{[i]} + \sum_{i=0}^{r} \hat{B}_{i,1}(I_m \otimes \hat{\bar{X}}_{s1}^{[i]}) U)$$

$$= \sum_{i=1}^{r} vec(\hat{A}_{i,1} \hat{\bar{X}}_{s1}^{[i]}) + \sum_{i=o}^{r} vec(\hat{B}_{i,1}(I_m \otimes \hat{\bar{X}}_{s1}^{[i]}) U)$$

$$(17)$$

and one can write

$$vec(\hat{A}_{i,1} \hat{\bar{X}}^{[i]}) = (\hat{\bar{X}}^{[i]T} \otimes I_{ns1}) vec(\hat{A}_{i,1}) \quad (18)$$

$$vec(\hat{B}_{i,1}(I_m \otimes \hat{\bar{X}}_{s1}^{[i]}) u) = \left[((I_m \otimes \hat{\bar{X}}_{s1}^{[i]}) u)^T \otimes I_{ns_1}\right] vec(\hat{B}_{i,1})$$

$$= \left[u^T (I_m \otimes \hat{\bar{X}}_{s1}^{[i]})^T \otimes I_{ns_1}\right] vec(\hat{B}_{i,1})$$

$$= (u^T \otimes I_{ns1})(I_m \otimes \hat{\bar{X}}_{s1}^{[i]T} \otimes I_{ns_1}) vec(\hat{B}_{i,1})$$

$$(19)$$

Considering equations (17), (18) and (19) we have then:

$$\dot{\bar{X}}_{s1} = \left[\bar{X}_{s1}^{\ T} \otimes I_{ns_1} \quad .. \quad \hat{\bar{X}}_{s1}^{\ [i]^T} \otimes I_{ns_1} \quad .. \right.$$

$$\left. u^T \otimes I \quad .. \quad (u^T \otimes I_{ns_1})(I_m \otimes \hat{\bar{X}}_{s1}^{\ [r]^T} \otimes I_{ns_1}) \right] \begin{bmatrix} vec(A_{1,1}) \\ \vdots \\ vec(\hat{A}_{i,1}) \\ \vdots \\ vec(\hat{B}_{0,1}) \\ \vdots \\ vec(\hat{B}_{r,1}) \end{bmatrix}$$

(20)

which can be written as a vectorial equation:

$$\dot{\bar{X}}_{s1} = \theta_{\bar{X}_{s1}} \vartheta_{s1} \qquad (21)$$

where

$$\theta_{\bar{X}_{s1}} = \left[\bar{X}_{s1}^{\ T} \otimes I_{ns_1} \quad .. \quad \hat{\bar{X}}_{s1}^{\ [i]^T} \otimes I_{ns_1} \quad .. u^T \otimes I \quad .. \quad (u^T \otimes I_{ns_1})(I_m \otimes \hat{\bar{X}}_{s1}^{\ [r]^T} \otimes I_{ns_1}) \right]$$

(22)

ϑ_{s1} is a vector of unknown coefficients defined by:

and
$$\vartheta_{s1} = \begin{bmatrix} vec(A_{1,1}) \\ \vdots \\ vec(\hat{A}_{i,1}) \\ \vdots \\ vec(\hat{B}_{0,1}) \\ \vdots \\ vec(\hat{B}_{r,1}) \end{bmatrix} \qquad (23)$$

So to derive the reduced order model, we have to determine the unknown vector ϑ_{s1} which needs the use of the Moore-Penrose pseudo-inverse $\theta_{\bar{X}_{s1}}^{\ +}$ given by:

$$\theta_{\bar{X}_{s1}}^{\ +} = \left[\theta_{\bar{X}_{s1}}^{T} \quad \theta_{\bar{X}_{s1}} \right]^{-1} \theta_{\bar{X}_{s1}} \qquad (24)$$

The solution of the vectorial equation (24) is then equal to:

$$\vartheta_{s1} = \theta_{\bar{X}_{s1}}^{\ +} \dot{\bar{X}}_{s1} \qquad (25)$$

Applying a random input to the system (15), the proposed approach provides the determination of the matrices $\hat{A}_{i,1}$ and $\hat{B}_{i,1}$, $i = 1, \cdots, r$ and so on the matrices $A_{i,1}$ and $B_{i,1}$ from (16).

Starting from the obtained reduced model (14) which contains $(q\text{-}1)$ state vectors of different speeds, the gradually application of the proposed approach of system reduction derives when going from the fastest to the slowest subsystem a reduced order slow subsystem model.

- After *(K +1)* stage of reduction

It appears from the above developments a recurrence. So, assumption 1 and 2 being verified at each stage, the successive separation procedure of order reduction leads after the kth stage to the following reduced model described by the polynomial state equations:

$$\dot{\bar{X}}_{sk} = \sum_{i=1}^{r} A_{i,k} \bar{X}_{sk}^{\ [i]} + \sum_{i=1}^{r} B_{i,k}(I_m \otimes \bar{X}_{sk}^{[i]}) U$$

(26)

where

$$\bar{X}_{sk} = [\bar{X}_{sk,1}^{\ T} \quad \bar{X}_{sk,2}^{\ T} \quad \cdots \bar{X}_{sk,j}^{\ T} \cdots \quad \bar{X}_{sk,q-k}^{\ T}]^T \in \mathfrak{R}^{n_{sk}}$$

and $\bar{X}_{sk,j} \in \mathfrak{R}^{n_j}$ with $n_{sk} = \sum_{j=1}^{q-k} n_j$.

After the calculus of the matrices $\hat{A}_{i,k}$ and $\hat{B}_{i,k}$ the next step is to extract again the fastest state variables given by $\bar{X}_{sk,q-k+1}$ and to derive the reduced model that approximate the slowest state variables of the full studied systems where the original variables are finally approximated by:

$$\bar{X}_{sk+1} = [\bar{X}_{sk+1,1}^{T} \quad \bar{X}_{sk+1,2}^{T} \quad \cdots \bar{X}_{sk+1,j}^{T} \cdots \quad \bar{X}_{sk+1,q-k}^{T}]^{T} \in \mathfrak{R}^{n_{s(k+1)}}$$

For this aim, it is necessary to develop the terms $\bar{X}_{sk}^{[i]}$ and $(I_m \otimes \bar{X}_{sk}^{[i]})$ (26) in order to make to appear the slow state variables $\bar{X}_{sk+1}$ and the fastest ones $\bar{X}_{sk,q-k}$.

Consequently, the system (26) can be formulated (appendix, [15]) by a similar description as (11) and (12).

It comes out that all the constant matrices $F_{ij,k}, G_{ij,k}, L_{ij,k}$ and $H_{ij,k}$ are then defined and we can derive the finally degenerate slow model:

$$\dot{\bar{X}}_{sk+1} = \sum_{i=1}^{r} A_{i,k+1} \bar{X}_{sk+1}^{[i]} + \sum_{i=1}^{r} B_{i,k+1} (I_m \otimes \bar{X}_{sk+1}^{[i]}) U$$

(27)

where $A_{i,1} = \hat{A}_{i,1} T_i^{+}$ and $B_{i,1} = \hat{B}_{i,1} (I_m \otimes T_i^{+})$

$\hat{A}_{i,k+1}$ and $\hat{B}_{i,k+1}$ are the solutions of the following equation: $\vartheta_{sk+1} = \theta_{\bar{X}_{sk+1}}^{+} \dot{\bar{X}}_{sk+1}$

ϑ_{sk+1} and $\theta_{\bar{X}_{sk+1}}^{+}$ are given by :

$$\theta_{\bar{X}_{sk+1}} = \left[\bar{X}_{sk+1}^{T} \otimes I_{ns_{k4}} \quad . \quad \hat{\bar{X}}_{sk+1}^{[r]^{T}} \otimes I_{ns_{k4}} \quad . \right.$$
$$\left. U^{T} \otimes I \quad .. \quad (U^{T} \otimes I_{ns_{k+1}})(I_m \otimes \hat{\bar{X}}_{sk+1}^{[r]^{T}} \otimes I_{ns_{k+1}}) \right]$$

(28)

and
$$\vartheta_{sk+1} = \begin{bmatrix} vec(A_{1,k+1}) \\ \vdots \\ vec(\hat{A}_{i,k+1}) \\ \vdots \\ vec(\hat{B}_{0,k+1}) \\ \vdots \\ vec(\hat{B}_{r,k+1}) \end{bmatrix}$$

(29)

and $\bar{X}_{sk+1}$ is the solution of the following equations:

$$\begin{cases} \dot{\bar{X}}_{sk+1} = F_{sk+1}(, \bar{X}_{q-k}) + G_{sk+1}(\bar{X}_{s1}, \bar{X}_{q-k}) U \\ 0 = H_{q-k}(\bar{X}_{sk+1}, \bar{X}_{q-k}) + L_{q-k}(\bar{X}_{sk+1}, \bar{X}_{q-k}) U \end{cases}$$

(30)

when computing, all the factor scale ε_i are incorporated in the corresponding matrices.

3.2 Illustrative Example

Let's consider a 7th order polynomial nonlinear system with four time-scales:

$$\begin{cases} \varepsilon_1 \dot{x}_1 = 1.6x_2 - 1.3x_3 + 0.3x_5 + 1.2x_6 - 0.15x_7 + 0.5x_1 x_5 \\ \quad + 0.5x_5^3 + 0.1x_6^3 + (1 - 0.5x_1 + 0.6x_2 + 0.5x_4 - 0.7x_5) u(t) \\ \varepsilon_1 \dot{x}_2 = -1.85x_1 - 0.95x_2 - 0.55x_3 + 0.15x_5 + 0.2x_6 - 0.2x_7 - 0.1x_3 x_5 \\ \quad + 0.15x_4^2 x_5 + 0.5x_2^2 - 0.2x_6 x_7 + (0.5 + 0.4x_1 + 0.7x_2 - 0.35x_4 - 0.2x_5) u(t) \\ \varepsilon_2 \dot{x}_3 = -0.39x_2 - 2.4x_3 - 0.1x_4 - 0.3x_5 - 0.09x_7 - 0.045x_1 x_4^2 + 0.3x_4 x_2 \\ \quad + (0.15 + 0.225x_1 x_4 + 0.06x_2^2 + 0.15x_4 - 0.15x_5) u(t) \\ \varepsilon_3 \dot{x}_4 = -0.211x_1 - 0.12x_2 - 0.16x_3 + 2.83x_4 - 0.06x_5 + 0.15x_6 \\ \quad + 0.3x_7 - x_4 x_5 + x_4^3 + (0.03 - 0.09x_2 + 0.12x_6^3 - 0.18x_5 x_1) u(t) \\ \varepsilon_3 \dot{x}_5 = -0.185x_1 - 0.1x_2 - 0.15x_3 + 0.26x_4 - 2.78x_5 - 0.16x_6 + x_3 x_5^2 \\ \quad + (0.15 - 0.075x_1 x_3 - 0.03x_4) u(t) \\ \varepsilon_4 \dot{x}_6 = -0.15x_1 - 0.165x_2 - 0.045x_3 - 0.3x_5 - 2.73x_6 - 0.345x_7 + 0.15x_1 x_5 \\ \quad - 0.45x_5^3 + (-0.009 + 0.04x_2^2 - 0.045x_3 x_4) u(t) \\ \varepsilon_4 \dot{x}_7 = -0.12x_1 - 0.198x_2 - 0.057x_3 - 0.3x_4 + 0.3x_6 - 2.7x_7 - 0.3x_4^2 x_5 \\ \quad - 0.4x_7^2 + (-0.045 - 0.027x_3 - 0.018x_4^2 + 0.009x_5 x_6) u(t) \end{cases}$$

After a three successive separation of dynamics, the proposed procedure provide the following slow reduced subsystem with second order described by

$$\dot{\overline{X}}_{s3} = A_{31}\,\overline{X}_{s3} + A_{32}\,\overline{X}_{s3}^{[2]} + A_{33}\,\overline{X}_{s3}^{[3]} + (\ B_{30} + B_{31}\ (I_m \otimes \overline{X}_{s3})$$
$$+ B_{32}\ (I_m \otimes \overline{X}_{s3}^{[2]}) + B_{33}\ (I_m \otimes \overline{X}_{s3}^{[3]}))\ U$$

where $\overline{X}_{s3} = \begin{bmatrix} \overline{X}_{s3,1} & \overline{X}_{s3,2} \end{bmatrix}^T$

When considering the random input signal as defined in Figure 1, all the characteristic matrices $A_{3,i}$ and $B_{3,i}$ are calculated.

$$A_{31} = \begin{bmatrix} 0.0509 & 1.5581 \\ -1.8042 & -0.9868 \end{bmatrix}, \qquad A_{32} = \begin{bmatrix} -0.1697 & -0.1161 & -0.1161 & 0.4522 \\ -0.0448 & 0.0005 & 0.0005 & 0.4064 \end{bmatrix}$$

$$A_{33} = \begin{bmatrix} 0.0898 & -0.0386 & -0.0386 & -0.1702 & -0.0386 & -0.1702 & -0.1702 & 0.2374 \\ 0.0315 & 0.0012 & 0.0012 & -0.0214 & 0.0012 & -0.0214 & -0.0214 & -0.2668 \end{bmatrix}$$

$$B_{30} = \begin{bmatrix} 0.8883 \\ 0.4828 \end{bmatrix}, B_{31} = \begin{bmatrix} -0.2513 & 0.4512 \\ 0.4448 & 0.8076 \end{bmatrix}, B_{32} = \begin{bmatrix} -0.4491 & 0.1403 & 0.1403 & 0.1746 \\ -0.0159 & -0.0582 & -0.0582 & 0.2182 \end{bmatrix}$$

$$B_{33} = \begin{bmatrix} 0.4441 & -0.1968 & -0.1968 & -0.4601 & -0.1968 & -0.4601 & -0.4601 & -0.4605 \\ 0.0945 & 0.0223 & 0.0223 & -0.1395 & 0.0223 & -0.1395 & -0.1395 & -0.3506 \end{bmatrix}$$

The obtained simulation results, Figure 2 to Figure 4, show the effectiveness of the recursive order reduction procedure and the quality of the reduced model as illustrate in the Figure 4.
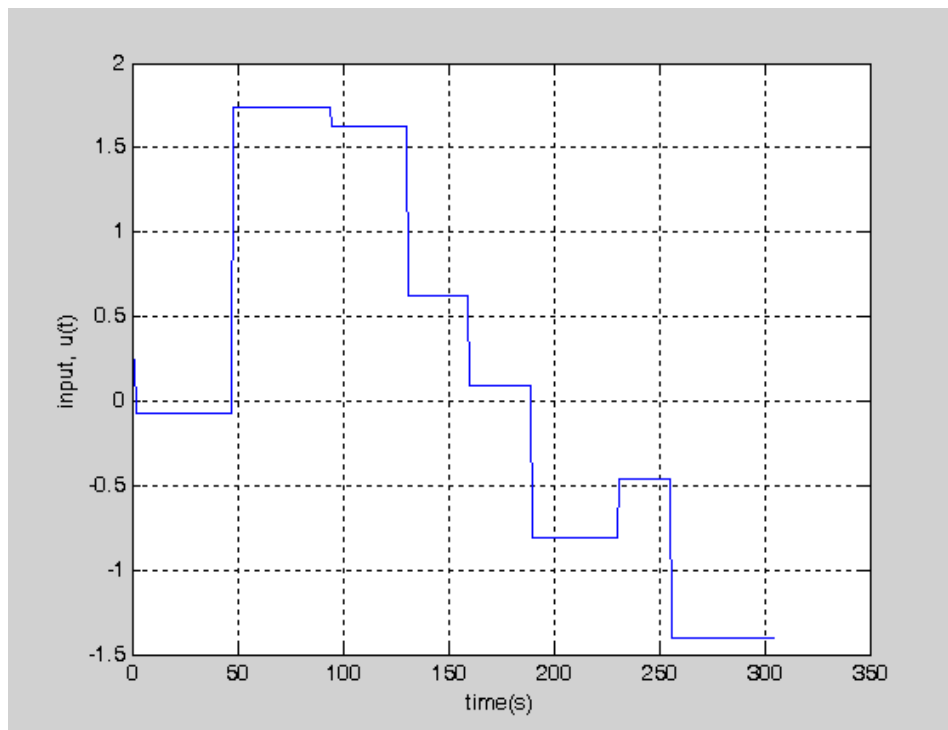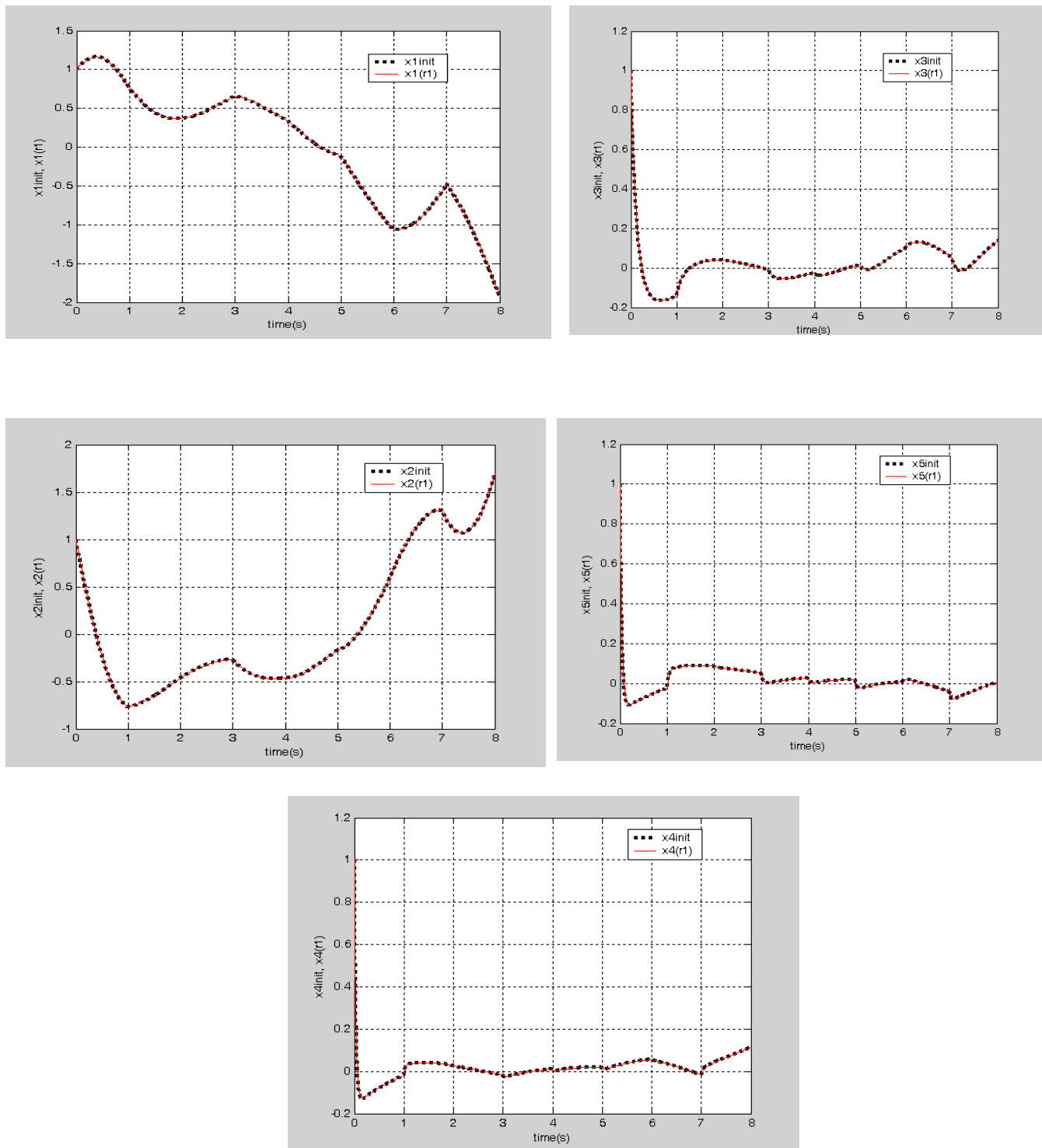

Figure 1. A random input signal u (t).

Figure 2. Simulation curve of state variables X i and obtained after the first step of reduction (___).
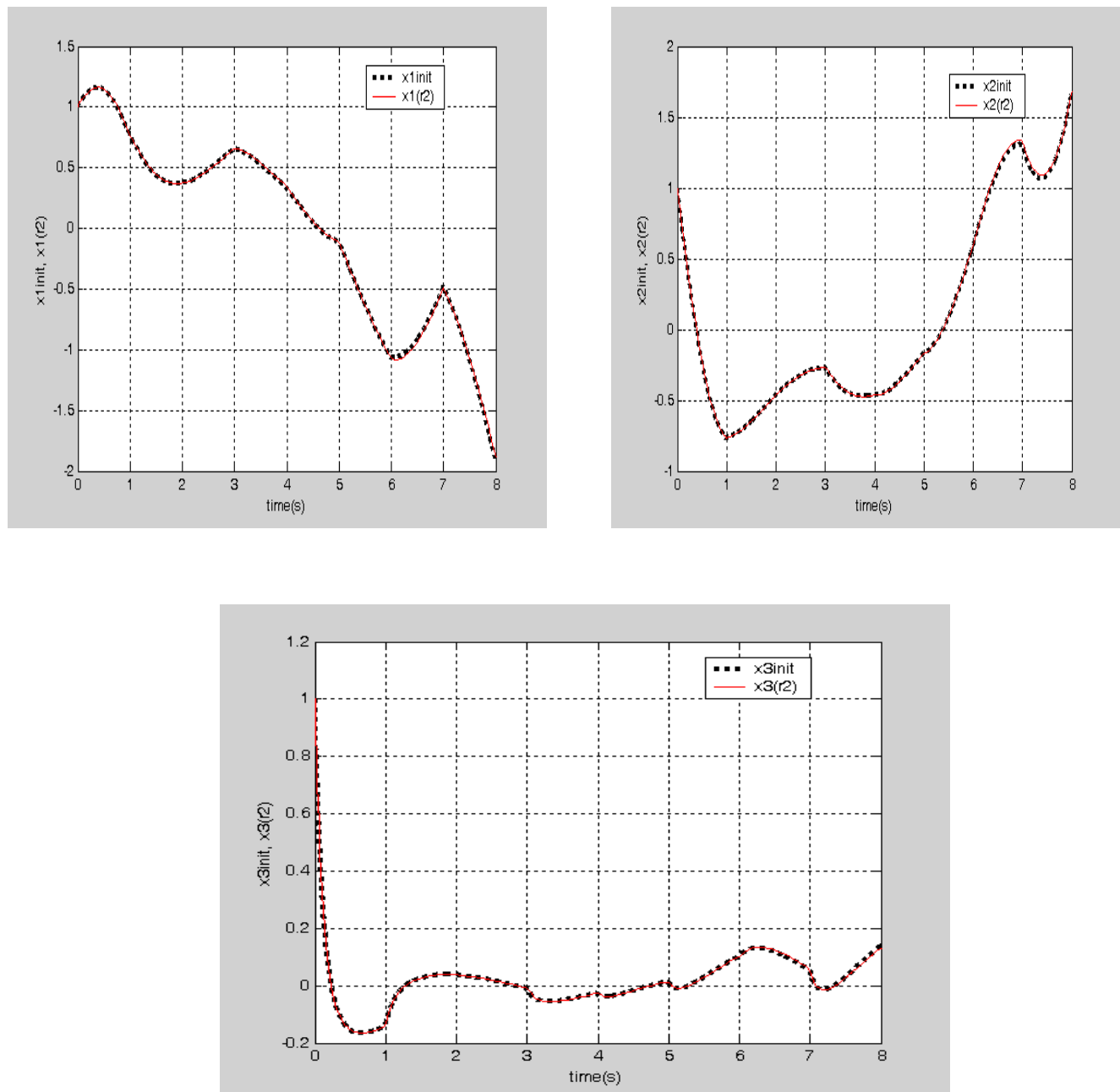
Figure 3. Simulation curve of state variables X i and obtained after the the second stage of reduction (____).
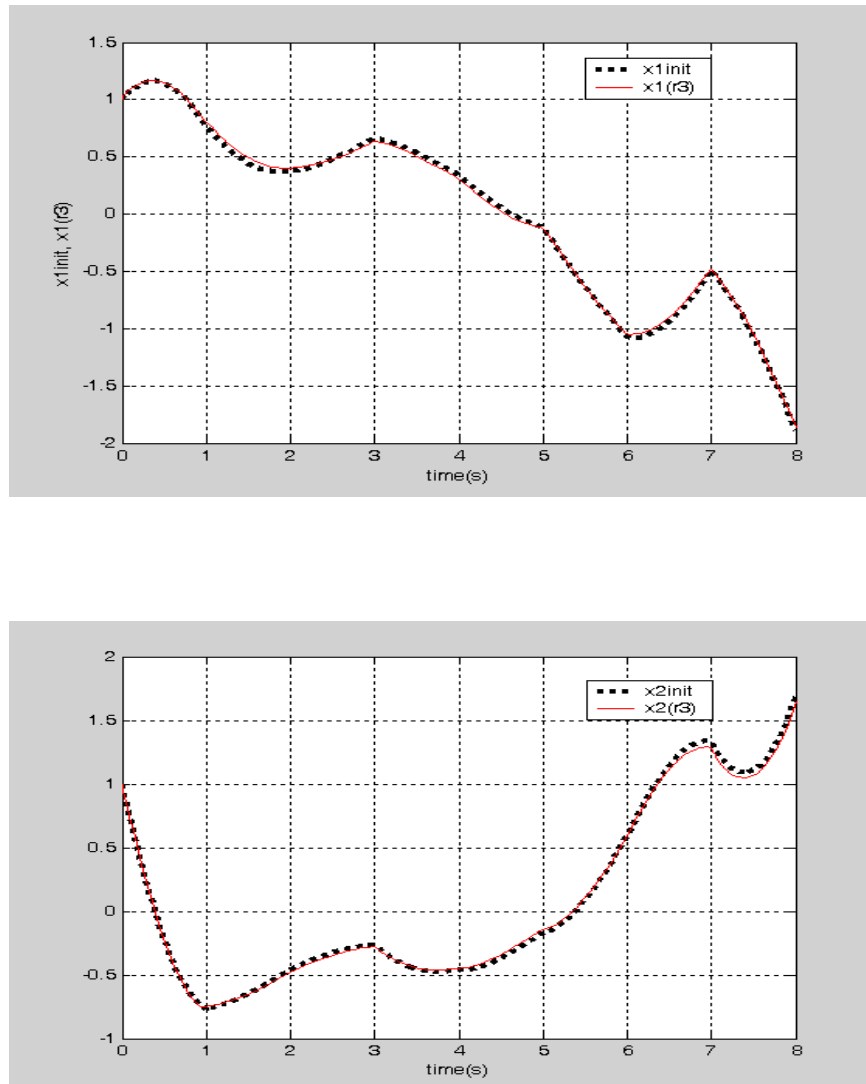
Figure 4. Simulation curve of state variables X i and (___) after three successive separation of dynamics.

Good results, figure 4, are obtained when comparing the state variables evolution of the initial system and those of the constructed reduced model showing the effectiveness of the proposed procedure.

4. Conclusion

The proposed approach appears to be a very powerful alternative to the general problem of modeling

complex systems and seems to correspond very well to industrial applications and especially to those interconnected with multi-time-scale. It relies on the use of the Kronecker product, polynomial representation and algebraic manipulations for the estimation of the matrices of the polynomial reduced order model. It must be pointed out that the algebraic formulation of the solutions of the nonlinear differential equations allows an easy implementation.

5. Appendix

The Kronecker product of A and B denoted by $A \otimes B$ is

defined by[9]:

$$A \otimes B = \begin{bmatrix} a_1 B & . & . & . & a_{1q} B \\ .a_2 B & . & . & . & a_{2q} B \\ . & & . & . & . \\ . & & . & . & . \\ a_{p1} B & . & . & . & a_p B \end{bmatrix} \quad (A.1)$$

$A \otimes B$ is then a ($pr \times qs$) matrix.

Among the main properties of this product[16], let us consider the following useful ones:

- Mixed product rules:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \quad (A.2)$$

- Transposition rule:

$$(A \otimes B)^T = A^T \otimes B^T \quad (A.3)$$

The Kronecker power of order i, noted $X^{[i]}$, of the

vector $X \in \Re^n$ is defined by:

$$\begin{cases} X^{[0]} = 1 \\ X^{[i]} = X^{[i-1]} \otimes X = X \otimes X^{[i-1]} \text{ pour } i \geq 1 \end{cases} \quad (A.4)$$

The nun-redundant i-power $\hat{X}^{[i]}$ of X is defined in[12]

as:

$$\hat{X}^{[1]} = X^{[1]} = X$$

$$\begin{cases} \forall i \geq 2; \ \hat{X}^{[i]} = [x_1^i, x_1^{i-1} x_2, ..., x_1^{i-1} x_n, x_1^{i-2} x_2^2, \\ \quad x_1^{i-2} x_2 x_3, ..., x_1^{i-2} x_2 x_n, ..., x_1^{i-2} x_n^2, ..., \ x_1^{i-3} x_2^3, ..., x_n^i]^T \end{cases}$$

$$(A.5)$$

It corresponds to the previous power where have been removed the repeated components.

Then we have:

$$\forall i \in \aleph; \ \exists \ !T_i \in \Re^{n^i \times \hat{n}_i}, \ \hat{n}_i = \binom{n+i-1}{i}; \ X^{[i]} = T_i \, \hat{X}^{[i]}$$

$$(A.6)$$

Thus, one possible solution for the inversion can be written as:

$$\forall \ i \in \aleph; \exists \ !T_i^+ \in \Re^{\hat{n}_i \times n^i} \ ; \ \hat{X}^{[i]} = T_i^+ \, X^{[i]} \quad (A.7)$$

where T_i^+ is the Moore-Penrose pseudo-inverse of T_i

given by:

$$T_i^+ = (T_i^T T_i)^{-1} T_i^T \quad (A.8)$$

An important vector valued function of a matrix was defined in[11] as follows:

$$\forall i \in \{1, ..., q\}, A_i \in \Re^p, A = \begin{bmatrix} A_1 & A_2 & \cdots & A_q \end{bmatrix};$$

$$vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{bmatrix} \quad (A.9)$$

We recall the following needed properties

$$vec(EAC) = (C^T \otimes E)vec(A) \qquad \text{(A.10)}$$

$$vec(A^T) = U_{p \times q} vec(A) \qquad \text{(A.11)}$$

$$vec(AC) = (C^T \otimes I_p)vec(A) = (I_g \otimes A)vec(C)$$

$$\text{(A.12)}$$

where

$U_{p \times q}$ is a square permutation matrix (pq x pq) defined

by :

$$U_{p \times q} = \sum_{i=1}^{p} \sum_{j=1}^{q} E_{i,j}^{p \times q} \otimes E_{i,j}^{p \times q} \qquad \text{(A.13)}$$

with $E_{i,j}^{p \times q} = e_i^p \otimes e_i^q$, where e_i^p is a unit p dimen-

sional vector which is '1' in the i-th column and zero elsewhere, and has precisely a single '1' in each row and in each column.

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