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More Results on Polygonal Sum Labeling of Graphs

Rajakishore Samal[1]* and Debdas Mishra[2]

[1]Department of Mathematics, O.P.S.Mohavidyalaya, Hindol Road, Dhenkanal – 759019, Odisha, India; rksamalmath@gmail.com

[2]Department of Mathematics, C.V.Raman College of Engineering and Technology, Bhubaneswar – 752054, Odisha, India; debdasmishra@gmail.com

Abstract

Objectives: To explore and identify some new classes of graphs which exhibit polygonal sum labeling. **Methods:** In this article we use a methodology which fundamentally involves formulation and subsequent mathematical validation.

Findings: Here we establish that the graphs – Star ($K_{1,n}$), Coconut Tree, Bistar ($S_{m,n}$), the Graph $S_{m,n,k}$, Comb (P_n K_1) and Subdivision graph $S\left(K_{1,n}\right)$ admit pentagonal, hexagonal, heptagonal, octagonal, nonagonal and decagonal sum labeling. **Applications:** One can explore to generalize these results and extend to give n-gonal labeling to some classes graphs. Sum labeling has already been used in the problems involving relational database management and hence one can try out to use polygonal sum labeling as well in these problems.

Keywords: Bistar, Coconut Tree, Comb, Polygonal Sum Labeling, Star, Subdivision Graph

1. Introduction

The graphs considered here are finite, connected, undirected and simple. The notations and terminologies involving graph theory may be found in[1] and the same involving number theory may be found in[2]. The study undertaken in this paper involves Polygonal sum labeling of graphs. A (p,q) graph G is said to admit a *polygonal sum labeling* if its vertices are labeled by non-negative integers such that the induced edge labels obtained by the sum of the labels of end vertices are the first q polygonal numbers. A graph possessing a polygonal sum labeling is called a *polygonal sum graph*. Here we show that some classes of graph can be embedded as induced sub graphs of a Polygonal sum graph. We recapitulate some important definitions useful for the present investigation.

1.1 Definition[3,4,5]

The numbers which generate a *k - gon* are known as *k-gonal* numbers. The n^{th} *k-gonal* (i.e. k - sided polygonal) number is given by $P_k(n) = \frac{n((k-2)n-k+4)}{2}$ where $k \geq 3$. For Example, The n^{th} pentagonal number is denoted by A_n and is given by the formula

$A_n = \frac{1}{2}n(3n-1)$. The few pentagonal numbers are $1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, \ldots$. Figure 1 illustrates pentagonal numbers.

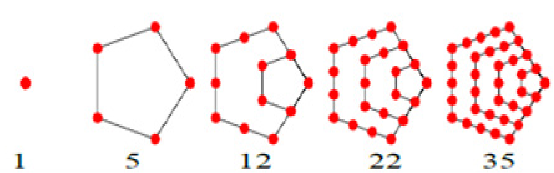

Figure 1. Pentagonal numbers 1, 5, 12, 22, 35.

1.2 Definition[3,5]

A *k - gonal* sum labeling of a graph G is a one to one function $f: V(G) \to N$ that induces a bijection

*Author for correspondence

$$f^{+}: E(G) \to \left\{ A_1, A_2 \ldots, A_q : A_n = \frac{n\left((k-2)n-k+4\right)}{2} \right\}$$

of the edges of G defined by $f^{+}(uv) \to f(u)+f(v)$ for every $e = uv \in E(G)$. The graph which admits such labeling is called a k - *gonal* sum graph.

1.3 Example

Figure 2 illustrates a pentagonal sum labeling of P_5.

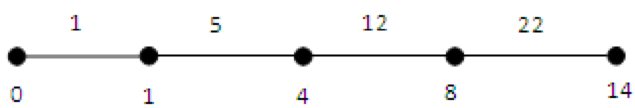

Figure 2. Pentagonal sum labeling of P_5.

1.4 Example

Figure 3 illustrates a decagonal sum labeling of P_{10}.

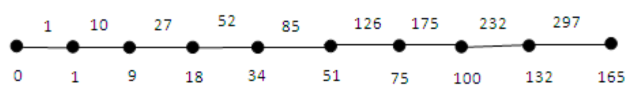

Figure 3. Decagonal sum labeling of P_5 .

In[5] give pentagonal, hexagonal, heptagonal, octagonal, nonagonal and decagonal sum labeling to paths. Amuthavalli and Dineshkumar[3] have given pentagonal sum labeling to bistars $S_{m,m}$. In this paper an attempt has been made to prove that the Star $K_{1,n}$, Coconut Tree, Bistar $S_{m,n}$, the Graph $S_{m,n,k}$, Comb $P_n \odot K_1$ and Subdivision graph $S(K_{1,n})$ admit pentagonal, hexagonal, heptagonal, octagonal, nonagonal and decagonal sum labeling. The graphs have been discussed in brief below.

1.5 Definition

Centre c with n pendant edges incident with c is called a Star graph and is denoted by $K_{1,n}$ or S_n. Hence it has $n+1$ vertices and n edges.

1.6 Definition

Coconut tree is a tree with central path $u_1, u_2, \ldots u_n$ having length $n-1$ and $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with u_1. Hence it has $n+k$ vertices and $n+k-1$ edges.

1.7 Definition

The graph obtained from $K_{1,m}$ and $K_{1,n}$ by joining their centers with an edge is called Bistar or Double star and is denoted by $S_{m,n}$. Let

$V(S_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and

$E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}$. Hence it has $m+n+2$ vertices and $m+n+1$ edges.

1.8 Definition

The graph $S_{m,n,k}$ is a graph obtained from a path of length k by attaching the stars $K_{1,m}$ and $K_{1,n}$ with its pendant vertices. Hence it has $m+n+k+1$ vertices and $m+n+k$ edges.

1.9 Definition

A graph obtained by attaching a single pendant edge to each vertex of a path $P_n = u_1 u_2 \ldots u_n$ is called a comb. A comb graph is obtained from the path by joining a vertex u_i to w_i, $1 \le i \le n$. It is denoted by $P_n \odot K_1$. The edges are labeled as $e_{2i-1} = u_i w_i$ and $e_{2i} = u_i u_{i+1}$ for $1 \le i \le n$. Hence it has $2n$ vertices and $(2n-1)$ edges.

1.10 Definition

The Subdivision of the star $K_{1,n}$ is a graph $S(K_{1,n})$ with vertex set $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \le i \le n\}$ and edge set $E(S(K_{1,n})) = \{vv_i, v_i u_i : 1 \le i \le n\}$. Hence it has $2n+1$ vertices and $2n$ edges.

2. Results

2.1 Pentagonal Sum Labeling of Graphs

In this section, we prove that stars S_n, coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \odot K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$

admit pentagonal sum labeling.

2.1.1 Theorem

The star graph $K_{1,n}$ or S_n possesses a pentagonal sum labeling.

Proof

Let u be the apex vertex and let $u_1, u_2, \ldots u_n$ be the pendant vertices of the star S_n. Define the labeling f by

$f(u) = 0$ and $f(u_i) = \frac{1}{2} i(3i-1), 1 \le i \le n.$ We see that the induced edge labels are the first n pentagonal numbers. Hence star graph $K_{1,n}$ or S_n possesses a pentagonal sum labeling.

2.1.2 Theorem

The coconut trees have pentagonal sum labeling.

Proof

Let $u_1, u_2, \ldots u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with u_1.

Define the labeling f by

$$f(u_i) = \begin{cases} \frac{1}{4}(i-1)(3i-1), & \text{if } i \text{ is odd} \\ \frac{1}{4} i(3i-4), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \le i \le n:$$

$$\text{and} \quad f(w_j) = \frac{1}{2}(n+j-1)(3n+3j-4), \quad \text{for } 1 \le j \le k.$$

We see that the induced edge labels are the first $n+k-1$ pentagonal numbers. Hence the coconut trees have pentagonal sum labeling.

2.1.3 Theorem

The bistar $S_{m,n}$ admits pentagonal sum labeling.

Proof

Let $V(S_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and

$$E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}$$

Define the labeling f by

$$f(u) = 0, f(v) = 1, f(u_i) = \frac{1}{2} i(3i-1), 1 \le i \le m,$$

$$\text{and } f(v_j) = \frac{1}{2}(m+j)(3m+3j-1) - 1, 1 \le j \le n.$$

We see that the induced edge labels are the first $m+n+1$ pentagonal numbers. Hence the bistar $S_{m,n}$ admits pentagonal sum labeling.

2.1.4 Theorem

The graphs $S_{m,n,k}$ admit pentagonal sum labeling.

Proof

Let $P_k : v_1, v_2, \ldots, v_{k+1}$ be a path of length k with initial vertex v_1 and terminal vertex v_{k+1}.

Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to v_1 and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to v_{k+1}.

Define the labeling f by

$$f(v_i) = \begin{cases} \frac{1}{4}(i-1)(3i-1), & \text{if } i \text{ is odd} \\ \frac{1}{4} i(3i-4), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \le i \le k,$$

$$f(u_j) = \frac{1}{2}(k+j)(3k+3j-1), \quad \text{for } 1 \le j \le m,$$

$$\text{and } f(w_l) = \frac{1}{2}(k+m+l)(3k+3m+3l-1) - f(v_{k+1}), \quad \text{for } 1 \le l \le n.$$

We see that the induced edge labels are the first $m+n+k$ pentagonal numbers. Hence the graphs $S_{m,n,k}$ admit pentagonal sum labeling.

2.1.5 Theorem

The comb $P_n \odot K_1$ possesses a pentagonal sum labeling.

Proof

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n-1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

For $i = 1, 2, \ldots, n,$ define the labeling f by

$$f(u_i)=\begin{cases}\frac{1}{4}(i-1)(3i-1), & \text{if } i \text{ is odd}\\ \frac{1}{4}i(3i-4), & \text{if } i \text{ is even}\end{cases} \text{ and } f(w_i)=\frac{1}{2}(n+i-1)(3n+3i-4)-f(u_i).$$

Thus the induced edge labels are the first $2n-1$ pentagonal numbers. Hence comb $P_n \odot K_1$ possesses a pentagonal sum labeling.

2.1.6 Theorem

$S(K_{1,n})$ the subdivision of the star graphs $K_{1,n}$ possesses a pentagonal sum labeling.

Proof

Let $V(S(K_{1,n}))=\{v, v_i, u_i : 1 \le i \le n\}$ and $E(S(K_{1,n}))=\{vv_i, v_iu_i : 1 \le i \le n\}$.

Define the labeling f by $f(v)=0$, $f(v_i)=\frac{1}{2}i(3i-1), 1 \le i \le n$, and $f(u_i)=\frac{1}{2}(n+i)(3n+3i-1)-f(v_i), 1 \le i \le n.$

We see that the induced edge labels are the first $2n$ pentagonal numbers and as such $S(K_{1,n})$ has a pentagonal sum labeling.

2.2 Hexagonal Sum Labeling of Graphs

In this section, we prove that star graphs S_n, coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \odot K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$ compatible with hexagonal sum labeling.

2.2.1 Theorem

The star graph $K_{1,n}$ or S_n has a hexagonal sum labeling.

Proof

Let u be the apex vertex and let $u_1, u_2, \ldots u_n$ be the pendant vertices of the star S_n.

Define the labeling f by $f(u)=0$ and $f(u_i)=i(2i-1), 1 \le i \le n.$

We see that the induced edge labels are the first n hexagonal numbers. Hence the star graph S_n has a hexagonal sum labeling.

2.2.2 Theorem

The coconut trees compatible with hexagonal sum labeling.

Proof

Let $u_1, u_2, \ldots u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with u_1.

Define the labeling f by $f(u_i)=\begin{cases}\frac{1}{2}(i-1)(2i-1), & \text{if } i \text{ is odd}\\ \frac{1}{2}i(2i-3), & \text{if } i \text{ is even}\end{cases}$ for $1 \le i \le n$

and $f(w_j)=(n+j-1)(2n+2j-3)$, for $1 \le j \le k$.

We see that the induced edge labels are the first $n+k-1$ hexagonal numbers and as such the coconut trees compatible with hexagonal sum labeling.

2.2.3 Theorem

The bistar $S_{m,n}$ admits hexagonal sum labeling.

Proof

Let $V(S_{m,n})=\{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and

$E(S_{m,n})=\{uv, uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}$.

Define the labeling f by

$f(u)=0$, $f(v)=1$, $f(u_i)=i(2i-1), 1 \le i \le m$,

and $f(v_j)=(m+j)(2m+2j-1)-1, 1 \le j \le n.$

We see that the induced edge labels are the first $m+n+1$ hexagonal numbers. Hence the bistar $S_{m,n}$ admits hexagonal sum labeling.

2.2.4 Theorem

The graph $S_{m,n,k}$ admits hexagonal sum labeling.

Proof

Let $P_k : v_1, v_2, \ldots, v_{k+1}$ be a path of length k with initial vertex v_1 and terminal vertex v_{k+1}. Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to v_1 and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to v_{k+1}.

Define the labeling f by $f(v_i)=\begin{cases}\frac{1}{2}(i-1)(2i-1), & \text{if } i \text{ is odd}\\ \frac{1}{2}i(2i-3), & \text{if } i \text{ is even}\end{cases}$ $\quad$ for $1\le i\le k+1$,

$f(u_j)=(k+j)(2k+2j-1)$, for $1\le j\le m$.

and $f(w_l)=(k+m+l)(2k+2m+2l-1)-f(v_{k+1})$, for $1\le l\le n$.

We see that the induced edge labels are the first $m+n+k$ hexagonal numbers. Hence the graph $S_{m,n,k}$ admits hexagonal sum labeling.

2.2.5 Theorem

The comb $P_n \odot K_1$ admits hexagonal sum labeling.
Proof

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n-1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

For $i = 1, 2, \ldots, n$, define the labeling f by

$$f(u_i)=\begin{cases}\frac{1}{2}(i-1)(2i-1), & \text{if } i \text{ is odd}\\ \frac{1}{2}i(2i-3), & \text{if } i \text{ is even}\end{cases} \quad \text{and} \quad f(w_i)=(n+i-1)(2n+2i-3)-f(u_i).$$

Thus the induced edge labels are the first $2n-1$ hexagonal numbers and as such the comb $P_n \odot K_1$ admits hexagonal sum labeling.

2.2.6 Theorem

$S(K_{1,n})$ the subdivision of the star $K_{1,n}$ admits hexagonal sum labeling.

Proof

Let $V(S(K_{1,n}))=\{v, v_i, u_i : 1\le i\le n\}$

and $E(S(K_{1,n}))=\{vv_i, v_iu_i : 1\le i\le n\}$.

Define the labeling f by
$f(v)=0$, $f(v_i)=i(2i-1)$, $1\le i\le n$, and

$f(u_i)=(n+i)(2n+2i-1)-f(v_i)$, $1\le i\le n$.

As the induced edge labels are the first $2n$ hexagonal numbers, $S(K_{1,n})$ admits hexagonal sum labeling.

2.3 Heptagonal Sum Labeling of Graphs

Here we prove that stars S_n , coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \odot K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$ admit heptagonal sum labeling.

2.3.1 Theorem

The star graph $K_{1,n}$ or S_n admits heptagonal sum labeling.
Proof

Let u be the apex vertex and let $u_1, u_2, \ldots u_n$ be the pendant vertices of the star S_n.

Define the labeling f by

$$f(u)=0 \text{ and } f(u_i)=\frac{1}{2}i(5i-3),\ 1\le i\le n.$$

We see that the induced edge labels obtained by the sum of the labels of the vertices are the first n heptagonal numbers. Hence star graph S_n admits heptagonal sum labeling.

2.3.2 Theorem

The coconut trees admit heptagonal sum labeling.
Proof

Let $u_1, u_2, \ldots u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with u_1. Define the labeling f by

$$f(u_i)=\begin{cases}\frac{1}{4}(i-1)(5i-3), & \text{if } i \text{ is odd}\\ \frac{1}{4}i(5i-8), & \text{if } i \text{ is even}\end{cases} \text{ for } 1\le i\le n \text{ and } f(w_j)=\frac{1}{2}(n+j-1)(5n+5j-8),\ \text{for } 1\le j\le k.$$

We see that the induced edge labels are the first $n+k-1$ heptagonal numbers. Hence the coconut trees admit heptagonal sum labeling.

2.3.3 Theorem

The bistar $S_{m,n}$ admit heptagonal sum labeling.

Proof

Let $V\left(S_{m,n}\right)=\left\{u,v,u_i,v_j:1\le i\le m,1\le j\le n\right\}$ and

$E\left(S_{m,n}\right)=\left\{uv,uu_i,vv_j:1\le i\le m,1\le j\le n\right\}$.

Define the labeling f by

$$f(u)=0,\ f(v)=1,\ f(u_i)=\frac{1}{2}i(5i-3),\ 1\le i\le m,$$

$$\text{and}\ \ f(v_j)=\frac{1}{2}(m+j)(5m+5j-3)-1,\ 1\le j\le n.$$

We see that the induced edge labels are the first $m+n+1$ heptagonal numbers. Hence bistar $S_{m,n}$ admit heptagonal sum labeling.

2.3.4 Theorem

The graph $S_{m,n,k}$ admits heptagonal sum labeling.

Proof

Let $P_k: v_1,v_2,\ldots,v_{k+1}$ be a path of length k with initial vertex v_1 and terminal vertex v_{k+1}. Let $u_1,u_2,\ldots,u_m$ be the adjacent vertices to v_1 and $w_1,w_2,\ldots,w_n$ be the adjacent vertices to v_{k+1} .

Define the labeling f by

$$f(v_i)=\begin{cases}\frac{1}{4}(i-1)(5i-3), & \text{if } i \text{ is odd}\\ \frac{1}{4}i(5i-8), & \text{if } i \text{ is even}\end{cases}\quad \text{for } 1\le i\le k+1,$$

$$f(u_j)=\frac{1}{2}(k+j)(5k+5j-3),\ \ \text{for}\ \ 1\le j\le m:$$

$$\text{and}\ \ f(w_l)=\frac{1}{2}(k+m+l)(5k+5m+5l-3)-f(v_{k+1}),\ \ \text{for}\ \ 1\le l\le n.$$

We see that the induced edge labels are the first $m+n+k$ heptagonal numbers. Hence the graph $S_{m,n,k}$ admits heptagonal sum labeling.

2.3.5 Theorem

The comb $P_n \odot K_1$ admits heptagonal sum labeling.

Proof

Let $P_n: u_1,u_2,\ldots,u_n$ be a path of length $n-1$ and let $w_1,w_2,\ldots,w_n$ be the pendant vertices adjacent to $u_1,u_2,\ldots,u_n$ respectively.

For $i=1,2,\ldots,n$: define

$$f(u_i)=\begin{cases}\frac{1}{4}(i-1)(5i-3), & \text{if } i \text{ is odd}\\ \frac{1}{4}i(5i-8), & \text{if } i \text{ is even}\end{cases}\quad \text{and}\quad f(w_i)=\frac{1}{2}(n+i-1)(5n+5i-8)-f(u_i).$$

Thus the induced edge labels are the first $2n-1$ heptagonal numbers. Hence comb $P_n \odot K_1$ admits heptagonal sum labeling.

2.3.6 Theorem

$S\left(K_{1,n}\right)$ the subdivision of the star $K_{1,n}$ admits heptagonal sum labeling.

Proof

Let $V\left(S\left(K_{1,n}\right)\right)=\left\{v,v_i,u_i:1\le i\le n\right\}$ and

$E\left(S\left(K_{1,n}\right)\right)=\left\{vv_i,v_iu_i:1\le i\le n\right\}$.

Define the labeling f by $f(v)=0,\ f(v_i)=\frac{1}{2}i(5i-3),\ 1\le i\le n$

$$\text{and}\ \ f(u_i)=\frac{1}{2}(n+i)(5n+5i-3)-f(v_i),\ 1\le i\le n.$$

We see that the induced edge labels are the first $2n$ heptagonal numbers. Hence $S\left(K_{1,n}\right)$ admits heptagonal sum labeling.

2.4 Octagonal Sum Labeling of Graphs

In this section, we prove that stars S_n , coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \odot K_1$, subdivision graphs $S\left(K_{1,n}\right)$ of the star $K_{1,n}$ admit octagonal sum labeling.

2.4.1 Theorem

The star graph $K_{1,n}$ or S_n admits octagonal sum labeling.

Proof

Let u be the apex vertex and let $u_1,u_2,\ldots u_n$ be the pendant vertices of the star S_n.

Define the labeling f by $f(u)=0$ and

$f(u_i)=i(3i-2),\ 1\le i\le n.$

We see that the induced edge labels obtained by the sum of the labels of the vertices are the first n octagonal numbers. Hence star graph S_n admits octagonal sum labeling.

2.4.2 Theorem

Coconut trees admit octagonal sum labeling.

Proof

Let $u_1, u_2, \ldots u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with u_1.

Define the labeling f by $f(u_i) = \begin{cases} \frac{1}{2}(i-1)(3i-2), & \text{if } i \text{ is odd} \\ \frac{1}{2}i(3i-5), & \text{if } i \text{ is even} \end{cases}$ for $1 \le i \le n$.

and $f(w_j) = (n+j-1)(3n+3j-5)$, for $1 \le j \le k$.

We see that the induced edge labels are the first $n+k-1$ octagonal numbers. Hence the coconut trees admit octagonal sum labeling.

2.4.3 Theorem

The bistar $S_{m,n}$ admits octagonal sum labeling.

Proof

Let $V(S_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and $E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}$.

Define the labeling f by $f(u) = 0$, $f(v) = 1$, $f(u_i) = i(3i-2)$, $1 \le i \le m$,
and $f(v_j) = (m+j)(3m+3j-2) - 1$, $1 \le j \le n$.

We see that the induced edge labels are the first $m+n+1$ octagonal numbers. Hence bistar $S_{m,n}$ admit octagonal sum labeling.

2.4.4 Theorem

The graph $S_{m,n,k}$ admits octagonal sum labeling.

Proof

Let $P_k : v_1, v_2, \ldots, v_{k+1}$ be a path of length k with initial vertex v_1 and terminal vertex v_{k+1}.

Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to v_1 and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to v_{k+1}.

Define the labeling f by

$$f(v_i) = \begin{cases} \frac{1}{2}(i-1)(3i-2), & \text{if } i \text{ is odd} \\ \frac{1}{2}i(3i-5), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \le i \le k+1,$$

$f(u_j) = (k+j)(3k+3j-2)$, for $1 \le j \le m$:

and $f(w_l) = (k+m+l)(3k+3m+3l-2) - f(v_{k+1})$, for $1 \le l \le n$.

We see that the induced edge labels are the first $m+n+k$ octagonal numbers. Hence the graph $S_{m,n,k}$ admits octagonal sum labeling.

2.4.5 Theorem

The comb $P_n \odot K_1$ admits octagonal sum labeling.

Proof

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n-1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively. For $i = 1, 2, \ldots, n$: define the labeling f by

$$f(u_i) = \begin{cases} \frac{1}{2}(i-1)(3i-2), & \text{if } i \text{ is odd} \\ \frac{1}{2}i(3i-5), & \text{if } i \text{ is even} \end{cases} \quad \text{and } f(w_i) = (n+i-1)(3n+3i-5) - f(u_i).$$

Thus the induced edge labels are the first $2n-1$ octagonal numbers. Hence comb $P_n \odot K_1$ admits octagonal sum labeling.

2.4.6 Theorem

$S(K_{1,n})$ the subdivision of the star $K_{1,n}$ admits octagonal sum labeling.

Proof

Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \le i \le n\}$ and

$$E(S(K_{1,n})) = \{vv_i, v_iu_i : 1 \le i \le n\}.$$

Define the labeling f by

$$f(v_i) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2}i(4i-7), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \le i \le k+1,$$

$f(u_j) = (k+j)(4k+4j-3)$, for $1 \le j \le m$,

and $f(w_l) = (k+m+l)(4k+4m+4l-3) - f(v_{k+1})$, for $1 \le l \le n$.

We see that the induced edge labels are the first $2n$ octagonal numbers. Hence S(K1,n) admits octagonal sum labeling.

2.5 Nonagonal Sum Labeling of Graphs

In this section, we prove that stars S_n, coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \odot K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$ admit nonagonal sum labeling.

2.5.1 Theorem

The star graph $K_{1,n}$ or S_n admits nonagonal sum labeling.

Proof

Let u be the apex vertex and let $u_1, u_2, \ldots u_n$ be the pendant vertices of the star S_n.

Define f by $f(u) = 0$ and $f(u_i) = \frac{1}{2} i(7i-5), 1 \le i \le n$.

We see that the induced edge labels obtained by the sum of the labels of the vertices are the first n nonagonal numbers. Hence star graph S_n admits nonagonal sum labeling.

2.5.2 Theorem

The coconut trees admit nonagonal sum labeling.

Proof

Let $u_1, u_2, \ldots u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with u_1.

Define the labeling f by

$$f(u_i) = \begin{cases} \frac{1}{4}(i-1)(7i-5), & \text{if } i \text{ is odd} \\ \frac{1}{4} i(7i-12), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \le i \le n \text{ and } f(w_j) = \frac{1}{2}(n+j-1)(7n+7j-12), \text{ for } 1 \le j \le k.$$

We see that the induced edge labels are the first $n+k-1$ nonagonal numbers. Hence Coconut trees admit nonagonal sum labeling.

2.5.3 Theorem

The bistar $S_{m,n}$ admit nonagonal sum labeling.

Proof

Let $V(S_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and $E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}$.

Define f by $f(u) = 0$, $f(v) = 1$, $f(u_i) = \frac{1}{2} i(7i-5), 1 \le i \le m$, and $f(v_j) = \frac{1}{2}(m+j)(7m+7j-5) - 1, 1 \le j \le n$.

We see that the induced edge labels are the first $m+n+1$ nonagonal numbers. Hence bistar $S_{m,n}$ admit nonagonal sum labeling.

2.5.4 Theorem

The graph $S_{m,n,k}$ admits nonagonal sum labeling.

Proof

Let $P_k : v_1, v_2, \ldots, v_{k+1}$ be a path of length k with initial vertex v_1 and terminal vertex v_{k+1}.

Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to v_1 and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to v_{k+1}.

Define f by $f(v_i) = \begin{cases} \frac{1}{4}(i-1)(7i-5), & \text{if } i \text{ is odd} \\ \frac{1}{4} i(7i-12), & \text{if } i \text{ is even} \end{cases}$ for $1 \le i \le k+1$,

$$f(u_j) = \frac{1}{2}(k+j)(7k+7j-5), \text{ for } 1 \le j \le m,$$

and $f(w_l) = \frac{1}{2}(k+m+l)(7k+7m+7l-5) - f(v_{k+1}), \text{ for } 1 \le l \le n.$

We see that the induced edge labels are the first $m+n+k$ nonagonal numbers. Hence the graph $S_{m,n,k}$ admits nonagonal sum labeling.

2.5.5 Theorem

The comb $P_n \odot K_1$ admits nonagonal sum labeling.

Proof

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n-1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

For $i = 1, 2, \ldots, n$: define the labeling f by

$$f(u_i) = \begin{cases} \frac{1}{4}(i-1)(7i-5), & \text{if } i \text{ is odd} \\ \frac{1}{4} i(7i-12), & \text{if } i \text{ is even} \end{cases} \text{ and } f(w_i) = \frac{1}{2}(n+i-1)(7n+7i-12) - f(u_i).$$

Thus the induced edge labels are the first $2n-1$ nonagonal numbers. Hence comb $P_n \odot K_1$ admits nonagonal sum labeling.

2.5.6 Theorem 2.5.6

$S(K_{1,n})$ the subdivision of the star $K_{1,n}$ admits nonagonal sum labeling.

Proof

Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \le i \le n\}$ and

$$E(S(K_{1,n})) = \{vv_i, v_iu_i : 1 \le i \le n\}.$$

Define the labeling f by $f(v)=0$, $f(v_i)=\frac{1}{2}i(7i-5), 1 \le i \le n,$

and $f(u_i)=\frac{1}{2}(n+i)(7n+7i-5)-f(v_i), 1 \le i \le n$.

We see that the induced edge labels are the first $2n$ nonagonal numbers. Hence $S(K_{1,n})$ admits nonagonal sum labeling.

2.6 Decagonal Sum Labeling of Graphs

In this section, we prove that stars S_n , coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \odot K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$ admit decagonal sum labeling.

2.6.1 Theorem

The star graph $K_{1,n}$ or S_n admits decagonal sum labeling.

Proof

Let u be the apex vertex and let $u_1, u_2, \ldots u_n$ be the pendant vertices of the star S_n.

Define f by $f(u)=0$ and

$$f(u_i)=i(4i-3), 1 \le i \le n.$$

We see that the induced edge labels obtained by the sum of the labels of the vertices are the first n decagonal numbers. Hence star graph S_n admits decagonal sum labeling.

2.6.2 Theorem

The coconut trees admit decagonal sum labeling.

Proof

Let $u_1, u_2, \ldots u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with u_1. Define the labeling f by

$$f(u_i)=\begin{cases}\frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd}\\ \frac{1}{2}i(4i-7), & \text{if } i \text{ is even}\end{cases} \quad for\ 1 \le i \le n \text{ and } f(w_j)=(n+j-1)(4n+4j-7), \ for\ 1 \le j \le k.$$

We see that the induced edge labels are the first $n+k-1$ decagonal numbers. Hence Coconut trees admit decagonal sum labeling.

2.6.3 Theorem 2.6.3

The bistar $S_{m,n}$ admit decagonal sum labeling.

Proof

Let $V(S_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and

$$E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}.$$

Define the labeling f by $f(u)=0$, $f(v)=1$, $f(u_i)=i(4i-3), 1 \le i \le m,$ and $f(v_j)=(m+j)(4m+4j-3)-1, 1 \le j \le n.$

We see that the induced edge labels are the first $m+n+1$ decagonal numbers. Hence bistar $S_{m,n}$ admit decagonal sum labeling.

2.6.4 Theorem

The graph $S_{m,n,k}$ admits decagonal sum labeling.

Proof

Let $P_k : v_1, v_2, \ldots, v_{k+1}$ be a path of length k with initial vertex v_1 and terminal vertex v_{k+1}.

Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to v_1 and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to v_{k+1} .

Define the labeling f by

$$f(v_i)=\begin{cases}\frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd}\\ \frac{1}{2}i(4i-7), & \text{if } i \text{ is even}\end{cases} \quad for\ 1 \le i \le k+1,$$

$$f(u_j)=(k+j)(4k+4j-3), \ for\ 1 \le j \le m,$$

and $f(w_l)=(k+m+l)(4k+4m+4l-3)-f(v_{k+1}), \ for\ 1 \le l \le n.$

We see that the induced edge labels are the first $m+n+k$ decagonal numbers. Hence the graph $S_{m,n,k}$ admits decagonal sum labeling.

2.6.5 Theorem 2.6.5

The comb $P_n \odot K_1$ admits decagonal sum labeling.

Proof

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n-1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

For $i = 1, 2, \ldots, n$: define the labeling f by

$$f(u_i) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2} i(4i-7), & \text{if } i \text{ is even} \end{cases} \text{ and } f(w_i) = (n+i-1)(4n+4i-7) - f(u_i).$$

Thus the induced edge labels are the first $2n-1$ decagonal numbers. Hence comb $P_n \odot K_1$ admits decagonal sum labeling.

2.6.6 Theorem

$S(K_{1,n})$ the subdivision of the star $K_{1,n}$ admits decagonal sum labeling.

Proof

Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \le i \le n\}$ and

$E(S(K_{1,n})) = \{vv_i, v_i u_i : 1 \le i \le n\}$.

and

Define the labeling f by $f(v) = 0$, $f(v_i) = i(4i-3)$, $1 \le i \le n$,

and $f(u_i) = (n+i)(4n+4i-3) - f(v_i)$, $1 \le i \le n$.

We see that the induced edge labels are the first $2n$ decagonal numbers. Hence $S(K_{1,n})$ admits decagonal sum labeling.

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