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Multiplicative Normed Linear Space and its Topological Properties

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Abstract

The aim of this article is to propose a new space called multiplicative normed linear space and discuss different topological properties of this space. We establish equivalence between multiplicative compactness and multiplicative sequentially compactness. Some examples are given which demonstrate the validity of our results. Further we apply our results to prove fixed point existence theorems.

Keywords: Multiplicative Continuous Function, Multiplicative Convergent Sequence, Fixed Point Existence Theorems, Multiplicative Normed Linear Space. **Mathematics Subject Classification** (2010): 46B20, 46B50, 47H10

1. Introduction

Bashirov et al[1] in 2008, studied the notion of multiplicative calculus and brought up multiplicative calculus to the attention of researchers in the branch of analysis and showed its usefulness. They defined a new space called multiplicative metric space. In 2012, Özavsar and Cevikel[2] discussed some topological properties of the multiplicative metric spaces. For more details of multiplicative metric spaces[2–4].

The concept of fixed point of mappings in metric as well as normed linear spaces is a powerful and important tool in nonlinear analysis, see references[6,8]. Also for important results concerning topological properties of normed linear spaces, the reader may refer to Kreyszig[5] and Lang[7].

Definition 1.1[1] Let X be a nonempty set. A multiplicative metric is a mapping $m : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(1) $m(x, y) \geq 1$ for all $x, y \in X$ and $m(x, y) = 1$ if and only if $x = y$;

(2) $m(x, y) = m(y, x)$ for all $x, y \in X$;

(3) $m(x, y) \leq m(x, z).m(z, y)$ for all $x, y, z \in X$.

The pair (X, m) is called a multiplicative metric space.

In this paper, we introduce the concept of multiplicative normed linear spaces. In section 2, we investigate topological properties of multiplicative normed linear spaces. In section 3, we prove some fixed point theorems in the context of multiplicative normed linear spaces.

Definition 1.2 Let X be a linear space over $\mathbb{R}$ (or $\mathbb{C}$). A mapping $\|.\|: X \to \mathbb{R}^+$ is called a multiplicative norm for X, if it satisfies the following conditions:

1. $\|x\| \geq 1$ for all $x \in X$;
2. $\|x\| = 1$ if and only if $x = 0$;
3. $\|\alpha x\| = \|x\|^{|\alpha|}$ for all $x \in X$ and any scalar α;
4. $\|x + y\| \leq \|x\|.\|y\|$ for all $x, y \in X$.

Then the pair $(X, \|.\|)$ is called a multiplicative normed linear space.

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It is clear by definitions of multiplicative normed linear spaces and normed linear spaces that both are independent spaces.

Example 1.3 Let $\mathbb{R}$ be the set of all real numbers. Let $||.||$ be defined on $\mathbb{R}$ by

$$||x|| = a^{|x|} \; ; \; x \in \mathbb{R}$$

where $a > 1$ is a fixed real number and

$$|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}.$$

Then $(||.||, \mathbb{R})$ is a multiplicative normed linear space.

Example 1.4 The space $C[a,b]$ of all continuous real-valued functions defined over $[a,b]$ is a multiplicative normed linear space under the multiplicative norm

$$||x|| = a^{\max_{t\in[a,b]}|x(t)|} \quad ,$$

where $x \in C[a,b]$ and $a > 1$ is a fixed real number.

Remark 1.5 Every multiplicative normed linear space $(X, ||.||)$ is a multiplicative metric space under the multiplicative metric given by $m(x,y) = ||x - y||$ for $x, y \in X$.

Proof: For any $x, y, z \in X$

(1) $m(x,y) = ||x - y|| \geq 1$ which implies $m(x,y) \geq 1$

(2) $m(x,y) = 1$ if and only if $||x - y|| = 1$ if and only if $x = y$

(3) $m(x,y) = ||x - y|| = ||y - x||^{|\ |} = m(y,x)$

(4) $m(x,y) = ||x - y + z - z|| \leq ||x - z||.||z - y|| = m(x,z).m(z,y)$

Thus (X,d) is a multiplicative metric space.

Lemma 1.6: A multiplicative metric m induced by a multiplicative norm $||.||$ on a multiplicative normed linear space X satisfies

(1) $m(x+a, y+a) = m(x,y)$

(2) $m(\alpha x, \alpha y) = m(x,y)^{|\alpha|}$

for any scalar α and $x, y, a \in X$.

Proof:

(1) $m(x+a, y+a) = ||x + a - y - a|| = ||x - y|| = m(x,y)$

(2) $m(\alpha x, \alpha y) = ||\alpha x - \alpha y|| = ||x - y||^{|\alpha|} = m(x,y)^{|\alpha|}$

Remark 1.7: Every multiplicative normed linear space is a multiplicative metric space but converse is not true, i.e. a multiplicative metric on a linear space may not be obtained from a norm.

Example 1.8: Let (X, m) be discrete multiplicative metric space defined as

$$m(x,y) = \begin{cases} 1 & if\ \ x = y \\ a & if\ \ x \neq y \end{cases}$$

where $a > 1$ is any fixed real number. Then it is a multiplicative metric space.

Suppose m is induced by a multiplicative norm $||.||$ on X, then

$$m(x,y) = ||x - y||$$

And by Lemma 1.6 $m(\alpha x, \alpha y) = m(x,y)^{|\alpha|}$ for any scalar α and $x, y \in X$.

Let $X = \mathbb{R}$ and take $\alpha = 3$, $x = 3$, $y = 2$, then $\alpha x \neq \alpha y$ which implies $m(\alpha x, \alpha y) = a$

Also $x \neq y$, so $m(x,y)^{|\alpha|} = a^3$

But $a \neq a^3$ for $a > 1$.

So this multiplicative metric cannot be induced by any multiplicative norm.

2. Topological Properties of Multiplicative Normed Linear Space

Definition 2.1 (Multiplicative ball): Let $(V, ||.||)$ be a multiplicative normed linear space, then given a point $x_0 \in V$, the multiplicative ball centered at x_0 and with radius $r > 1$ is the set

$$B_r(x_0)=\{v\in V \,|\, \| v-x_0 \|<r\}.$$

Definition 2.2 (Multiplicative Open set): Let $(V,\|.\|)$ be a multiplicative normed linear space. A set $G\subset V$ is called multiplicative open if for every $x\in G$, there exists $\varepsilon>1$ such that

$$B_\varepsilon(x)\subset G.$$

Definition 2.3 (Multiplicative Closed set): Let $(V,\|.\|)$ be a multiplicative normed linear space. A set $F\subset V$ is multiplicative closed if $F^c=V-F$ is multiplicative open.

Definition 2.4 (Convergence in multiplicative normed linear space): A sequence $\{x_n\}_{n=1}^{\infty}$ in a multiplicative normed linear space $(V,\|.\|)$ is said to be multiplicative convergent to $x\in V$, if for any $\varepsilon>1$, there exists $n_0\in\mathbb{N}$ such that

$$\| x_n-x \|<\varepsilon \text{ for all } n\geq n_0$$

or simply we can say $x_n \to_* x$ as $n\to\infty$ if $\lim_{n\to\infty}\| x_n-x \|=1$.

Definition 2.5 (Multiplicative Cauchy sequence): A sequence $\{x_n\}_{n=1}^{\infty}$ in a multiplicative normed linear space $(V,\|.\|)$ is said to be multiplicative Cauchy sequence, if for any $\varepsilon>1$, there exists $n_0\in\mathbb{N}$ such that

$$\| x_m-x_n \|<\varepsilon \text{ for all } m,n\geq n_0$$

it is denoted by $\| x_m-x_n \|\to_* 1$ as $m,n\to\infty$.

Definition 2.6 (Bounded set in multiplicative normed linear space): Let $(V,\|.\|)$ be a multiplicative normed linear space. A set $S\subset V$ is said to be bounded, if there exists a constant M such that

$$\| x \|\leq M \text{ for all } x\in S.$$

Lemma 2.7: Let $(X,\|.\|)$ be a multiplicative normed linear space.

(i) Every multiplicative convergent sequence is multiplicative Cauchy sequence in X.

(ii) If $x_n \to_* x$ and $x_n \to_* y$ then $x=y$ i.e. multiplicative limits are unique.

(iii) Every multiplicative Cauchy sequence in X is bounded.

Proof: (i) Let $x\in X$ such that $x_n \to_* x$.

Then for any $\varepsilon>1$, there exists a natural number n_0 such that $\| x_n-x \|<\sqrt{\varepsilon}$ and $\| x_m-x \|<\sqrt{\varepsilon}$ for all $n,m\geq n_0$. Then by multiplicative triangle inequality, we get

$$\| x_m-x_n \|=\| x_m-x+x-x_n \|\leq\| x_m-x \|.\| x_n-x \|$$
$$<\sqrt{\varepsilon}.\sqrt{\varepsilon}=\varepsilon \quad \text{for all } m,n\geq n_0$$

which implies $\{x_n\}_{n=1}^{\infty}$ is a multiplicative Cauchy sequence.

(ii) $\| x-y \|=\| x-x_n+x_n-y \|$

$\leq\| x_n-x \|.\| x_n-y \|\to_* 1$ as $n\to\infty$ which implies $\| x-y \|=1$ i.e. $x=y$

(iii) Let $\{x_n\}_{n=1}^{\infty}$ be a multiplicative Cauchy sequence in X. Take $\varepsilon=2$, then by definition of multiplicative Cauchy sequence there exists a natural number n_0 such that

$$\| x_n-x_m \|<2 \text{ for all } m,n\geq n_0$$

In particular $\| x_n-x_{n_0} \|<2$

which implies, for all $n\geq n_0$,

$$\begin{aligned}\| x_n \| &=\| x_n-x_{n_0}+x_{n_0} \|\\ &\leq\| x_n-x_{n_0} \|.\| x_{n_0} \|\\ &<2.\| x_{n_0} \|\end{aligned}$$

Then $\| x_n \|\leq\max(\| x_1 \|,\| x_2 \|,\ldots,\| x_{n_0} \|,2\| x_{n_0} \|)$ for all $n\in\mathbb{N}$.

Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

Remark 2.8: Since every multiplicative convergent sequence is multiplicative Cauchy sequence and every multiplicative Cauchy sequence is bounded. So every multiplicative convergent sequence is bounded.

Definition 2.9: Let $(X, \|.\|)$ be a multiplicative normed linear space and let $A \subseteq X$.

(i) A point $x \in X$ is called *multiplicative limit point* of A, if there exists a sequence $x_n \in A$ with $x_n \neq x$ such that $x_n \rightarrow_* x$.

(ii) A is *multiplicative closed*, if it contains all its multiplicative limit points. In other words, A is multiplicative closed, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of elements of A and $x_n \rightarrow_* x \in X$, then x must be the element of A.

(iii) The set $\overline{A} = A \bigcup A'$ is called *multiplicative closure of the set* A, where A' is the set of all multiplicative limits points of A.

(iv) We say that A is *multiplicative dense* in X if multiplicative closure of A is equal to A i.e. $\overline{A} = A$.

Definition 2.10 (Multiplicative Banach space): A multiplicative normed linear space $(X, \|.\|)$ is called multiplicative complete if every multiplicative Cauchy sequence in X multiplicative convergent to a limit in X. A complete multiplicative normed linear space is called a multiplicative Banach space.

Example 2.11: $\mathbb{R}^n$ is multiplicative Banach space under the multiplicative norm defined by

$$\|x\| = a^{\sum_{i=1}^{n}|x_i|},$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $a > 1$ is a fixed real number.

Solution: First we shall show that $\mathbb{R}^n$ is a multiplicative normed linear space.

(i) $\|x\| = a^{\sum_{i=1}^{n}|x_i|} \geq 1 \ as \sum_{i=1}^{n}|x_i| \geq 0$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$;

(ii) $\|x\| = a^{\sum_{i=1}^{n}|x_i|} = 1$ if and only if $\sum_{i=1}^{n}|x_i| = 0$ if and only if $x = (x_1, x_2, \ldots, x_n) = 0$;

(iii) $\|\alpha x\| = a^{\sum_{i=1}^{n}|\alpha x_i|} = a^{|\alpha|\sum_{i=1}^{n}|x_i|} = \|x\|^{|\alpha|}$ for any scalar α and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$;

(iv) $\|x + y\| = a^{\sum_{i=1}^{n}|x_i + y_i|}$

$\leq a^{(|x_1|+|x_2|+\ldots+|x_n|)} . a^{(|y_1|+|y_2|+\ldots+|y_n|)} = \|x\| . \|y\|$

for any $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Thus $\mathbb{R}^n$ is a multiplicative normed linear space under the above defined multiplicative norm.

Now we shall show that $\mathbb{R}^n$ is multiplicative complete.

Let $\{x_m\}_{m=1}^{\infty}$ be any multiplicative Cauchy sequence in $\mathbb{R}^n$, where $x_m = (x_1^m, x_2^m, \ldots, x_n^m) \in \mathbb{R}^n$.

Let $\varepsilon > 1$ be given, then by definition of multiplicative Cauchy sequence, there exists an integer $n_0 > 0$ such that

$\varepsilon > \|x_m - x_l\|$ for all $m, l \geq n_0$

$= a^{|x_1^m - x_1^l| + |x_2^m - x_2^l| + \ldots + |x_n^m - x_n^l|}$ for all $m, l \geq n_0$ ...(2.1)

$> a^{|x_i^m - x_i^l|}$ for all $m, l \geq n_0$ and $i = 1, 2, \ldots, n$,

which is true for any given $\varepsilon > 1$, when m and l are taken greater than some positive integer. Therefore, we have

$|x_i^m - x_i^l| \rightarrow 0 \quad as \quad m, l \rightarrow \infty$

Thus $\{x_i^m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ with usual norm for all $i = 1, 2, \ldots, n$. But $\mathbb{R}$ is complete with usual norm. Therefore there exists $z_i \in \mathbb{R}$ such that

$$\lim_{m\to\infty} x_i^m = z_i \quad \text{for all } i = 1, 2, \ldots, n$$

Now in equation (2.1), letting $l \to \infty$, we get

$$\varepsilon > a^{\sum_{i=1}^{n}\left|x_i^m - \lim_{l\to\infty} x_i^l\right|} \quad \text{for all } m \geq n_0$$

$$= a^{\sum_{i=1}^{n}\left|x_i^m - z_i\right|} \quad \text{for all } m \geq n_0$$

$$= \| x_m - z \| \quad \text{for all } m \geq n_0,$$

where $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$

Which implies $\{x_m\}_{m=1}^{\infty}$ is multiplicative convergent to $z \in \mathbb{R}^n$.

Hence $\mathbb{R}^n$ is multiplicative Banach space under the above defined multiplicative norm.

Theorem 2.12: A subspace Y of a multiplicative Banach space X is multiplicative complete if and only if Y is multiplicative closed in X.

*Proof***:** Let Y be multiplicative closed subspace of multiplicative Banach space X. Let $\{x_n\}$ be any multiplicative Cauchy sequence in Y. This implies $\{x_n\}$ is multiplicative Cauchy sequence in X. But X is multiplicative complete, therefore $\{x_n\}$ multiplicative converges to some element x in X. Since Y is multiplicative closed, we conclude that $x \in Y$. Thus $\{x_n\}$ multiplicative converges to some element of Y. Hence Y is multiplicative complete.

Conversely, we assume that Y is multiplicative complete subspace of multiplicative Banach space X. We shall prove that Y is multiplicative closed in X i.e. $\overline{Y} = Y$. Since $Y \subset \overline{Y} = Y \cup Y'$, so it is sufficient to prove that $\overline{Y} \subset Y$. Let $x \in \overline{Y}$ then there exists a sequence $\{x_n\}$ in Y which multiplicative converges to x. So $\{x_n\}$ is multiplicative Cauchy sequence in Y. But Y is multiplicative complete, so $\{x_n\}$ multiplicative converges in Y. Using Lemma 2.7, we have $x \in Y$.

Definition 2.13 (Multiplicative Continuous function): Let $(X, \|\,.\,\|_1)$ and $(Y, \|\,.\,\|_2)$ be two multiplicative normed linear spaces and $f : X \to Y$ be a function. We say that f is multiplicative continuous at $x_0 \in X$, if for any given $\varepsilon > 1$, there exists $\delta > 1$ such that $\| x - x_0 \|_1 < \delta$ implies $\| f(x) - f(x_0) \|_2 < \varepsilon$.

Theorem 2.14: Let $(X, \|\,.\,\|_1)$ and $(Y, \|\,.\,\|_2)$ be two multiplicative normed linear spaces. A mapping $f : X \to Y$ is continuous at a point $x_0 \in X$ if and only if $x_n \to_* x_0$ in X implies $f(x_n) \to_* f(x_0)$ in Y.

*Proof***:** Let f be continuous at x_0. Then for a given $\varepsilon > 1$, there exists $\delta > 1$ such that

$$\| x - x_0 \|_1 < \delta \text{ implies } \| f(x) - f(x_0) \|_2 < \varepsilon .$$

Now let $x_n \to_* x_0$ in X. Then by definition of multiplicative convergent sequence, there exists a natural number n_0 such that

$$\| x_n - x_0 \|_1 < \delta \text{ for all } n \geq n_0 .$$

Which implies, $\| f(x_n) - f(x_0) \|_2 < \varepsilon$ for all $n \geq n_0$.

Therefore by definition $f(x_n) \to_* f(x_0)$ in Y.

Conversely, we assume that $x_n \to_* x_0$ in X implies $f(x_n) \to_* f(x_0)$ in Y.

We shall prove that f is continuous at x_0. Suppose this is not true, then there exists an $\varepsilon > 1$ such that for every $\delta > 1$, we have x in X other than x_0 satisfying

$$\| x - x_0 \|_1 < \delta \text{ but } \| f(x) - f(x_0) \|_2 \geq \varepsilon .$$

In particular, for $\delta_n = 1 + \dfrac{1}{n}$, we have x_n in X satisfying

$$\| x_n - x_0 \|_1 < 1 + \frac{1}{n} \text{ but } \| f(x_n) - f(x_0) \|_2 \geq \varepsilon$$

which implies $x_n \rightarrow_* x_0$ but $\{f(x_n)\}$ does not multiplicative converge to $f(x_0)$.Which is contradiction. Hence f is continuous at x_0.

Proposition 2.15: Let $(X, \|.\|_1)$ and $(Y, \|.\|_2)$ be two multiplicative normed linear spaces. Then the Cartesian product $X \times Y = \{(x,y); x \in X, y \in Y\}$ is also a multiplicative normed linear space with the following norms

(1) $\|(x,y)\| = \|x\|_1 . \|y\|_2$

(2) $\|(x,y)\|_\infty = \max\{\|x\|_1, \|y\|_2\}$

Proof: (1) First we shall show that Cartesian product $X \times Y$ is a multiplicative normed linear space under the multiplicative norm

$$\|(x,y)\| = \|x\|_1 . \|y\|_2$$

(i) $\|(x,y)\| = \|x\|_1 . \|y\|_2 \geq 1$,

since $\|x\|_1 \geq 1, \|y\|_2 \geq 1$ for all $(x,y) \in X \times Y$.

(ii) $\|(x,y)\| = 1$ if and only if $\|x\|_1 . \|y\|_2 = 1$

if and only if $\|x\|_1 = 1 = \|y\|_2$ if and only if $x = 0 = y$

(iii) $\|\alpha(x,y)\| = \|(\alpha x, \alpha y)\| = \|\alpha x\|_1 . \|\alpha y\|_2 = \|x\|_1^{|\alpha|} . \|y\|_2^{|\alpha|} = (\|x\|_1 . \|y\|_2)^{|\alpha|}$

$= \|(x,y)\|^{|\alpha|}$ for all $(x,y) \in X \times Y$ and any scalar α.

(iv) Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, then

$$\|(x_1,y_1) + (x_2,y_2)\| = \|(x_1 + x_2, y_1 + y_2)\| = \|x_1 + x_2\|_1 . \|y_1 + y_2\|_2$$

$$\leq \|x_1\|_1 . \|x_2\|_1 . \|y_1\|_2 . \|y_2\|_2 = \|(x_1,y_1)\| . \|(x_2,y_2)\|$$

(2)Weshallshowthat $\|(x,y)\|_\infty = \max\{\|x\|_1, \|y\|_2\}$ is multiplicative norm on $X \times Y$.

(i) $\|(x,y)\|_\infty = \max\{\|x\|_1, \|y\|_2\} \geq 1$

since $\|x\|_1 \geq 1, \|y\|_2 \geq 1$ for all $(x,y) \in X \times Y$.

(ii) $\|(x,y)\|_\infty = 1$ if and only if $\max\{\|x\|_1, \|y\|_2\} = 1$

if and only if $\|x\|_1 = 1 = \|y\|_2$ if and only if $x = 0 = y$

(iii) $\|\alpha(x,y)\|_\infty = \|(\alpha x, \alpha y)\|_\infty = \max\{\|\alpha x\|_1, \|\alpha y\|_2\} = \max\{\|x\|_1^{|\alpha|}, \|y\|_2^{|\alpha|}\}$

$$= (\max\{\|x\|_1, \|y\|_2\})^{|\alpha|} = \|(x,y)\|_\infty^{|\alpha|}.$$

for all $(x,y) \in X \times Y$ and any scalar α.

(iv) Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, then

$$\|(x_1,y_1) + (x_2,y_2)\|_\infty = \|(x_1 + x_2, y_1 + y_2)\|_\infty = \max\{\|x_1 + x_2\|_1, \|y_1 + y_2\|_2\}$$

$$\leq \max\{\|x_1\|_1 . \|x_2\|_1, \|y_1\|_2 . \|y_2\|_2\}$$

$$\leq \max\{\|x_1\|_1, \|y_1\|_2). \max\{\|x_2\|_1, \|y_2\|_2\}$$

$$= \|(x_1,y_1)\|_\infty . \|(x_2,y_2)\|_\infty$$

Theorem 2.16: Let $(V, \|.\|)$ be a multiplicative normed linear space over the scalar field F. Then

(i) The mapping $(\alpha, x) \rightarrow \alpha x$ from $F \times V \rightarrow V$ is multiplicative continuous.

(ii) The mapping $(x,y) \rightarrow x + y$ from $V \times V \rightarrow V$ is multiplicative continuous.

(iii) The mapping $x \rightarrow \|x\|$ from $V \rightarrow \mathbb{R}$ is multiplicative continuous i.e. multiplicative norm is multiplicative continuous.

Proof: (i) Let $\alpha_n \rightarrow \alpha$ in F and $x_n \rightarrow_* x$ in V.

We shall show that $\alpha_n x_n \rightarrow_* \alpha x \ as \ n \rightarrow \infty$.

Consider $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$

$$\leq \|\alpha_n(x_n - x)\| . \|(\alpha_n - \alpha)x\|$$

$$= \|x \quad - x\|^{|\alpha_n|} \quad \|x\|^{|\alpha_n \quad \alpha|}$$

Since $|\alpha_n - \alpha| \rightarrow 0$ and $\|x_n - x\| \rightarrow_* 1$ as $n \rightarrow \infty$

we obtain $\|\alpha_n x_n - \alpha x\| \rightarrow_* 1 \ as \ n \rightarrow \infty$

i.e. $\alpha_n x_n \rightarrow_* \alpha x \ as$ as $n \rightarrow \infty$

(ii) Let $x_n \rightarrow_* x$, $y_n \rightarrow_* y$ as $n \rightarrow \infty$

$\| x_n - x \| \to_* 1$ and $\| y_n - y \| \to_* 1$ as $n \to \infty$

Consider $\| (x_n + y_n) - (x + y) \| = \| (x_n - x) + (y_n - y) \|$

$\leq \| x_n - x \| . \| y_n - y \| \to_* 1 \;\; as \;\; n \to \infty$

i.e. $(x_n + y_n) \to_* (x + y) \;\; as \;\; n \to \infty$

(iii) Before proving (iii), we first obtain an inequality

$$\frac{\| y \|}{\| x - y \|} \leq \| x \| \leq \| x - y \| . \| y \| \quad .$$

We note that in a multiplicative normed linear space

$$\| x \| = \| x - y + y \| \leq \| x - y \| . \| y \| \tag{2.2}$$

Interchanging the role of x and , we obtain

$$\| y \| \leq \| y - x \| . \| x \| \quad . \tag{2.3}$$

Using (2.2) and (2.3), we obtain

$$\frac{\| y \|}{\| x - y \|} \leq \| x \| \leq \| x - y \| . \| y \| \quad . \tag{2.4}$$

Taking $x = x_n, y = x$ in equation (2.4), we get

$$\Rightarrow \frac{\| x \|}{\| x_n - x \|} \leq \| x_n \| \leq \| x_n - x \| . \| x \| \quad . \tag{2.5}$$

Now let $x_n \to_* x,$ then by equation (2.5), we get

$$\| x_n \| \to_* \| x \|$$

Thus multiplicative norm is multiplicative continuous function.

Definition 2.17 (Multiplicative Compact set): Let $(X, \| . \|)$ be a multiplicative normed linear space and let A be a subset of X. Then A is said to be multiplicative compact set, if every multiplicative open cover of A has a finite subcover i.e. A is multiplicative compact, whenever $\{A_\alpha\}_{\alpha \in I}$ is a collection of multiplicative open sets whose union contains A, then there exist finitely many $\alpha_1, \alpha_2, \ldots, \alpha_N$

such that $A \subseteq \bigcup_{i=1}^{N} A_{\alpha_i}$.

Definition 2.18 (Multiplicative Sequentially Compact set): Let $(X, \| . \|)$ be a multiplicative normed linear space and let A be a subset of . Then A is said to be multiplicative sequentially compact set, if every sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of A contains a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ which multiplicative converges to some element of A.

Definition 2.19 (Multiplicative totally bounded): Let $(X, \| . \|)$ be a multiplicative normed linear space and let A be a subset of X. Then A is said to be multiplicative totally bounded, if for any $\varepsilon > 1,$ there exists finitely many points $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B_\varepsilon(a_i)$ where $B_\varepsilon(a_i)$ is the multiplicative open ball of radius ε and centre a_i.

Lemma 2.20 : Any multiplicative closed subset of a multiplicative compact normed space X is multiplicative compact.

Proof: Let A be a multiplicative closed subset of a multiplicative compact normed space X. Let $\{A_\alpha\}_{\alpha \in I}$ be a multiplicative open cover of A. Then $\{A_\alpha\}_{\alpha \in I} \cup (X - A)$ is a multiplicative open cover of X and therefore X has a finite subcover say $\mathbf{S}$. Then $\mathbf{S} - \{X - A\}$ is a finite subset of $\{A_\alpha\}_{\alpha \in I}$ that covers A.

Theorem 2.21: In a multiplicative normed linear space, every multiplicative compact set is a multiplicative sequentially compact set.

Proof: Let A be a multiplicative compact subset of a multiplicative normed linear space X. Suppose that A is not sequentially compact, then there exists a sequence $\{a_n\}$ in A that does not have a multiplicative convergent subsequence. Consider the set $S = \{a_1, a_2, \ldots\}$. Then clearly $S=\overline{S}$, since if for some $x\in\overline{S}, x\notin S$, in this case $B_{\varepsilon}(x)\cap S\neq\phi$ for all $\varepsilon>1$, then we can construct some subsequence in S converging multiplicatively to x. Thus S is multiplicatively closed subset of a multiplicative compact set A. So S is compact (by Lemma 2.20). As $\{a_n\}$ does not have multiplicative convergent subsequence, for any $m\in\mathrm{N}$, there exists $\varepsilon_m>1$ such that $B_{\varepsilon_m}(a_m)=a_m$. But $\{B_{\varepsilon_m}(a_m)\}_{m=1}^{\infty}$ is a multiplicative open cover of S and so it has a finite subcover. But this implies atleast one term in the sequence $\{a_n\}$ must be infinitely repeated, which is contradiction.

In order to prove multiplicative sequentially compactness implies multiplicative compactness, we need two lemmas.

Lemma 2.22: Every multiplicative sequentially compact subset of a multiplicative normed linear space X is totally bounded.

Proof: Suppose it is not true. Then there exists a set A which is multiplicative sequentially compact, but for some $\varepsilon_0>1$, there exists no finite set S such that $A\subset\bigcup_{x\in s}B_{\varepsilon_0}(x)$. Now choose an element $x_1\in A$. There exists some $x_2\in A-B_{\epsilon_0}(x_1)$. Similarly we can find some $x_3\in A-(B_{\epsilon_0}(x_1)\cup B_{\epsilon_0}(x_2))$ and so on. Constructing a sequence in such a way, we have $x_n\in A$ such that $\|x_i-x_j\|>\epsilon_0$ for all $i\neq j$. Therefore $\{x_n\}$ cannot have any convergent subsequence, which is a contraction.

Lemma 2.23: Let A be a multiplicative sequentially compact subset of a multiplicative normed linear space X and $\mathbf{S}$ is an multiplicative open cover of A. Then there exists an $\varepsilon>1$ such that for any $x\in A$ we can find an $S_x\in\mathbf{S}$ such that $B_{\varepsilon}(x)\subset S_x$.

Proof: Suppose it is not true, then for any $\varepsilon_m=1+\frac{1}{m}$ where $m\in\mathbb{N}$, there exists $x_m\in A$ such that $B_{\varepsilon_m}(x_m)$ is not contained in any element of $\mathbf{S}$. Since A is multiplicative sequentially compact, we can find a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ that multiplicative converges to some $x\in A$. Now $\mathbf{S}$ is multiplicative open cover of A and $x\in A$ so $x\in S^*$ for some $S^*\in\mathbf{S}$. As S^* is multiplicative open, we have $B_{\varepsilon}(x)\subset S^*$ for some $\varepsilon>1$. Since $\{x_{m_k}\}$ converges multiplicatively to x, we have some $N\in\mathbb{N}$ such that $x_{m_k}\in B_{\varepsilon}(x)$ for all $k>N$. Also $B_{\varepsilon}(x)$ is open, therefore there exist $\varepsilon_{m_k}>1$ such that $B_{\varepsilon_{m_k}}(x_{m_k})\subset B_{\varepsilon}(x)$. Thus $B_{\varepsilon_{m_k}}(x_{m_k})\subset S^*\in\mathbf{S}$, which is contraction.

Theorem 2.24: Every multiplicative sequentially compact set is multiplicative compact in a multiplicative normed linear space X.

Proof: Let A be a multiplicative sequentially compact set and let $\mathbf{S}$ be an open cover of A. By lemma 2.23, there exists an $\varepsilon_0>1$ such that for any $x\in A$ we can find $S_x\in\mathbf{S}$ such that $B_{\varepsilon_o}(x)\subset S_x$. Also by lemma 2.22, for $\varepsilon_0>1$ there exists a finite subset B of A such that $A\subset\bigcup_{x\in B}B_{\varepsilon_0}(x)$. Thus $A\subset\bigcup_{x\in B}B_{\varepsilon_0}(x)\subset\bigcup_{x\in B}S_x$ where $S_x\in\mathbf{S}$ and hence A is compact.

By Theorems 2.21 and 2.24, we can conclude that in a multiplicative normed linear space, multiplicative compactness is equivalent to multiplicative sequentially compactness.

3. Some Fixed Point Theorems in Multiplicative Normed Linear Spaces

Theorem 3.1: Let T be a continuous mapping on a Multiplicative Banach space X. If there exists $x,y\in X$ such that $\{T^n(x)\}_{n=1}^{\infty}$ multiplicative converges to y, then y is fixed point for T.

Proof: Given that T is continuous and $T^{n-1}(x) \to_* y$ as $n \to \infty$.

Therefore $T(T^{n-1}(x)) \to_* T(y)$ as $n \to \infty$. i.e. $\| T^n(x) - T(y) \| \to_* 1$ as $n \to \infty$.

Now consider,

$$\| T(y) - y \| = \| T(y) - T^n(x) + T^n(x) - y \|$$

$$\leq \| T(y) - T^n(x) \| . \| T^n(x) - y \| \to_* 1 \text{ as } n \to \infty.$$

i.e. $T(y) = y$.

Theorem 3.2: Let   be a continuous mapping on a multiplicative Banach space X. If $T(X)$ is a compact set in X and for each   1, there exists a $x \in X$ such that $\| T(x) - x \| < \varepsilon$. Then T has fixed point.

Proof: By given condition, for each $\varepsilon_n = 1 + \frac{1}{n}$, there exists $x_n \in X$ such that

$$\| T(x_n) - x_n \| < \varepsilon_n \qquad (3.1)$$

Since $T(X)$ is compact set in X, there exists a subsequence $\{T(x_{n_k})\}_{k=1}^{\infty}$ of $\{T(x_n)\}_{n=1}^{\infty}$ which multiplicative converges to some point of X ,say x .

$$\text{i.e. } \| T(x_{n_k}) - x \| \to_* 1 \text{ as } k \to \infty. \qquad (3.2)$$

Now using (3.1), we have

$$\| x_{n_k} - x \| = \| x_{n_k} - x + T(x_{n_k}) - T(x_{n_k}) \|$$

$$\leq \| x_{n_k} - T(x_{n_k}) \| . \| T(x_{n_k}) - x \|$$

$$< \in_{n_k} . \| T(x_{n_k}) - x \| \to_* 1 \text{ as } k \to \infty.$$

i.e. $\| x_{n_k} - x \| \to_* 1$ as $k \to \infty$.

Since T is continuous, and $x_{n_k} \to_* x$,

We have $T(x_{n_k}) \to_* T(x)$, …(3.3) Using (3.2) and (3.3),

we obtain $T(x) = x$.

4. Conclusion

In this paper, we have introduced important tools like multiplicative ball, multiplicative open set and multiplicative closed set in multiplicative normed linear spaces. Also boundedness of multiplicative Cauchy sequence has been proved. Continuity of scaler multiplication, vector addition and multiplication norm have been proved in multiplicative normed linear spaces. Further we have concluded that in a multiplicative normed linear space, multiplicative compactness is equivalent to multiplicative sequentially compactness. Moreover, applications of topological properties of multiplicative normed linear spaces to prove fixed point existence theorems have been given.

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6. References

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