ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645
Indian Journal of Science and Technology, *Vol 10(1), DOI: 10.17485/ijst/2017/v10i1/109977, January 2017*

Wavelet-Collocation Method of Solving Singular Integral Equation

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Abstract

Objectives: This article describes one of the approaches to the approximate solution of the singular integral equation of the first kind with the Cauchy kernel on material axis interval, based on the approximation of the desired function by Chebyshev's wavelets of the II-nd kind. **Methods**: The theory of such equations states that the solution in a closed form can be obtained only in rare cases. Therefore, various approximation methods are used, followed by their theoretical basis. The Uniform error estimates of obtained approximate solutions are very relevant ones for practice. **Results**: However, the incorrect problem of this equation solution caused primarily by universal values of first and reversible singular operators, on the pair of continuous function spaces, leads to particular difficulties at a numerical solution. The work as the area of desired elements and right sides considers weighted spaces, some of which are the restrictions of continuous functions on which a correct task is set. A computational scheme of wavelet collocation method is developed. The theorem on the unique solvability of obtained linear algebraic equation system is proved. Uniform estimates for the relative solution errors are set depending on the structural properties of original data. **Conclusion**: The performed numerical experiment in Wolfram Mathematical package showed a real convergence of the approximate solution, obtained by the method of wavelet collocation, with the exact one.

Keywords: Chebyshev's Wavelets, Singular Integral Equation, Wavelet Collocation Method

1. Introduction

Let us consider the singular integral equation (i.e.) of the first kind with Cauchy kernel:

$$Kx \equiv \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}x(\tau)}{\tau - t}d\tau + \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}h(t,\tau)x(\tau)d\tau = f(t), \quad |t|<1, \quad (1)$$

Where $h(t,\tau)$, $f(t)$ are known continuous functions in their ranges of definition, $x(\tau)$ isthe required function and the singular integral

$$I\varphi = I(\varphi;t) = \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(\tau)}{\tau - t}d\tau$$

Is understood in the sense of Cauchy principal value.

The equation (1) is widely used in many fields of science and technology. The theory of such equations is well developed[1]. According to this theory the obtaining of i.e. exact solution (1) in a closed form is possible only in certain cases. Therefore, various approximate methods of its solution are developed, which are based on the idea of the desired solution in the form of classical orthogonal polynomials[2].

It is known that the problem of the equation (1) solution is an incorrectly posed in many pairs of functional spaces, including the pair of continuous function spaces. However, there is the possibility of finding a cor-

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rect formulation of a problem for equation solution (1). The work[3] sets the correctness of the specified equation problem solution (1) on a pair of weighted spaces (X, Y), where X is the space of continuous functions $x(t)$ on [-1; 1], for which the singular integral $I(\rho x; t)$ is also a continuous function, $\rho \equiv \rho(t) = \sqrt{1-t^2}$, and Y is the space of continuous functions $f(t)$ on [-1; 1] such, that $\frac{1}{q(t)} I(qf;t)$ is the continuous function on (-1; 1), which allows a continuous extension on the ends of the segment, $q \equiv q(t) = \frac{1}{\sqrt{1-t^2}}$ and the following term is performed

$$\int_{-1}^{1} q(t)y(t)dt = 0 \cdot \tag{2}$$

The norms in these areas are determined respectively in the following way:

$$\|x\|_X = \|\rho x\|_C + \|I(\rho x)\|_C, \; x \in X;$$

$$\|f\|_Y = \|f\|_C + \left\|\frac{1}{q} I(qf)\right\|_C, \; f \in Y,$$

Where $\|x\|_C = \max_{-1 \le t \le 1} |x(t)|$.

Then i.e. (1) is equivalent to operator equation

$$Kx \equiv Sx + Vx = f \quad (x \in X, f \in Y), \tag{3}$$

$$Sx = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\tau^2}\,x(\tau)}{\tau - t} d\tau, \; Vx = \frac{1}{\pi}\int_{-1}^{1} \sqrt{1-\tau^2}\, h(t,\tau) x(\tau) d\tau \cdot$$

This pair of spaces carried out[3] the substantiation of various approximation methods based on the idea of the approximate solution in the form of polynomials. At the end of the last century, a new class of basic functions named wavelets appeared. Nowadays, the wavelet theory is being actively developed, finds its application in various fields of science[4,5], while leaving a vast field for research. In recent years, the theory of wavelets obtained an intensive development in the works of many authors based on trigonometric and algebraic polynomials, as well as the methods of function expansion in series according to polynomial wavelets [6-11]. Wavelet has been widely used to solve image processing problems including classification by extracting better features from each image[12-14]. In adopted the concept of integral form to solve partial differential equation in mechanical problem[15-20]. The collocation method was applied in this paper to solve the singular integral equation (1). The method was based on the approximation of the desired function by Chebyshev's wavelets of the II-nd kind.

2. Wavelet Collocation Method

Let's consider the "regularized" equation

$$\overline{K}\overline{x} \equiv Sx + Vx + \gamma = f \quad (\overline{x} \in \overline{X}, f \in Y, \gamma \in R), \tag{4}$$

Where $\overline{X} = (X, \gamma)$ with the norm

$$\|\overline{x}\|_{\overline{X}} = \|x\|_X + |\gamma|,$$

$\overline{x} = (x, \gamma)$ is a vector function with the following components $x(t) \in X$ and $\gamma \in R$.

The equation (2) shows that the regularization parameter is the following one:

$$\gamma = \int_{-1}^{1} \frac{f(t) - V(x;t)}{\sqrt{1-t^2}} dt \cdot$$

An approximate solution of the equation (1) will be sought in the form of the vector function

$$\overline{x_m} = (x_m, \gamma_m), \; x_m(t) = a_0\varphi_{0,0}(t) + a_1\varphi_{0,1}(t) + \sum_{j=0}^{m-1}\sum_{k=0}^{2^j-1} b_{j,k}\psi_{j,k}(t), \tag{5}$$

Where

$$\varphi_{m,k}(t) = \sum_{j=0}^{2^m} U_j(t) U_j\left(\xi_k^{(2^m+1)}\right) \frac{2\left|\sin\frac{\pi(k+1)}{2^m+2}\right|}{\sqrt{\pi(2^m+2)}}, \quad m = 0,1,\ldots, \quad k = \overline{0,2^m},$$

$$\psi_{m,k}(t) = \sum_{j=2^m+1}^{2^{m+1}} U_j(t) U_j\left(\xi_k^{(2^m)}\right) \frac{2\left|\sin\frac{\pi(k+1)}{2^m+1}\right|}{\sqrt{\pi(2^m+1)}}, \quad m = 1,2,\ldots, \quad k = \overline{0,2^m-1}$$

The so-called scaling function and Chebyshev's wavelet function of the II-nd kind, respectively[21], $U_j(t) = \frac{\sin(j+1)\arccos t}{\sqrt{1-t^2}}, \; j = 0,1,2,\ldots$. are Chebyshev's polynomial of the II-nd kind, $\xi_k^{(n)} = \cos\frac{\pi(k+1)}{n+1}, \; k = 0,\ldots,n-1$ are polynomial zeros $U_n(t)$, $\gamma_m \in R$.

The unknown coefficients a_0, a_1, $b_{j,k}$ ($j = \overline{0,m-1}, k = \overline{0,2^j-1}$), γ_m will be sought from the condition of discrepancy zero equality in the nodes of collocation

$$t_k^{(2^m+2)} = \cos\frac{\pi\left(k+\frac{1}{2}\right)}{2^m+2},\ m = 1,2,\ldots,k = \overline{0,2^m+1}. \tag{6}$$

Taking into account the known[2] ratios $SU_n = -T_{n+1}$, $T_n(t) = \cos n \arccos t$ -Chebyshev's polynomials of the I-st kind, we obtain the system of linear algebraic equations

$$\gamma_m + \sum_{j=0}^{1} T_{j+1}(t_k^{(2^m+2)})\left(-a_0\frac{U_j(\xi_0^{(2)})}{\sqrt{\pi}} - a_1\frac{U_j(\xi_1^{(2)})}{\sqrt{\pi}}\right) -$$

$$-\sum_{j=0}^{m-1}\sum_{k=0}^{2^j-1}\sum_{i=2^j+1}^{2^{j+1}} b_{j,k}T_{i+1}(t_k^{(2^m+2)})U_i(\xi_k^{(2^j)})\frac{2\left|\sin\frac{\pi(k+1)}{2^j+1}\right|}{\sqrt{\pi(2^j+1)}} + \tag{7}$$

$$+\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}h(t_k^{(2^m+2)},\tau)\left(a_0\varphi_{0,0}(\tau) + a_1\varphi_{0,1}(\tau) + \sum_{j=0}^{m-1}\sum_{k=0}^{2^j-1} b_{j,k}\psi_{j,k}(\tau)\right)d\tau = f(t_k^{(2^m+2)}),\ k = \overline{0,2^m+1}.$$

Let's denote the set of functions with a continuous derivative of r-th order, satisfying Holder's condition with the value α, 0<α ≥1, r^3 0 via W^rH_α = W^rH [-1, 1].

Theorem: Let's the following terms are satisfied:

a) the equation (4) has a single solution $\overline{x_m^*} \in \overline{X}$ at any right part $f \in Y$;

б) the function $f(t) \in W^r H_\alpha$, the kernel $h(t,\tau) \in W^r H_\alpha$ according to variable t is uniform relative to τ.

Then starting from some $m \in N$, the system of collocation method (7) has a single value $\grave{a}_0^*$, $\grave{a}_1^*$, $b_{j,k}^*$ ($j = \overline{0,m-1}, k = \overline{0,2^j-1}$), γ_m^* and the approximate solutions $\overline{x_m^*} = (x_m^*, \gamma_m^*)$ are converged to exact solutionx*in the areaXwith the speed

$$\left\|\overline{x}^* - \overline{x_m^*}\right\|_{\tilde{O}} = \left(\frac{\ln(2^m+1)}{(2^m+1)^{r+\alpha}}\right),\ 0<\alpha\le 1,\ r\ge 0.$$

Proving: Let's H_{2^m+1} is the set of all algebraic polynomials of the degree not exceeding 2^m+1. Let's introduce the subspaces of vector functions $\overline{X_m} = \{\overline{x}_m\}$, $\overline{x_m} = (x_m, \gamma_m)$, $x_m \in H_{2^m+1}$, $\gamma_m \in R$ with the following norm $\|\overline{x}_m\|_{\overline{X_m}} = \|x_m\|_{H_{2^m+1}} + |\gamma_m|$; $Y_m = H_{2^m+2}$. Then the system of the collocation method may be written in an operator form as follows:

$$\overline{K}_m\overline{x}_m \equiv \gamma_m + Sx_m + L_mVx_m = L_mf, \tag{8}$$

$$\left(\overline{x}_m \in \overline{X_m},\ L_mf \in H_{2^m+1}\right),$$

Where $L_mf : Y \to H_{2^m+1}$ is the operator, which assigns the correspondence of continuous function f with Lagrange interpolation poly nomial according to node system (6).

We obtain the following one from (4) and (8) for any $\overline{x}_m \in \overline{X_m}$

$$\left\|\left(\overline{K} - \overline{K_m}\right)\overline{x}_m\right\|_Y = \|Vx_m - L_mVx_m\|_Y \le \|h - L_m^t h\|_{Y\otimes C}\|x_m\|_X,$$

where the operator L_m^t is applied to the function $h(t;\tau)$ according to variable t.

From results[3] and theorem conditions we obtain the following one:

$$\overline{\varepsilon}_n \equiv \left\|\left(\overline{K} - \overline{K_m}\right)\overline{x}_m\right\|_{\overline{X_m}\to Y} = O\left(\frac{\ln(2^m+1)}{(2^m+1)^{r+\alpha}}\right) \to 0,\ m\to\infty. \tag{9}$$

Under the conditions of the theorem the operator $\overline{K} : \overline{X} \to Y$ is continuously reversible. Then due to (9) and the following ratios

$$\overline{\delta}_n \equiv \|f - L_mf\|_Y = O\left(\frac{\ln(2^m+1)}{(2^m+1)^{r+\alpha}}\right) \to 0,\ m\to\infty$$

The required statement follows from the theorem 7 of the chapter I[22].

3. Numerical Experiment

Let's consider the following singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}x(\tau)}{\tau - t}d\tau + \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}(t+\tau^3)x(\tau)d\tau = -t^5 - \frac{3}{2}t^3 - \frac{33}{16}t,\quad |t|<1. \tag{10}$$

The exact equation (10) at that

$$x^*(\tau) = \tau^4 + 2\tau^2 + 7.$$

The search of an approximate solution according to the above-stated calculation scheme implemented in Wolfram Mathematic.

Let's m = 1. The solution of equation (10) under the scheme of the wavelet collocation (7) will be the following one is shown in Figures 1 and 2.

$$\overline{x_1^*} = \left(x_1^*, \gamma_1^*\right),$$

Where

$$x_1^*(t) = \frac{0,625\sin 3\arccos t}{\sqrt{1-t^2}} + 7,66,\ \gamma_1^* = 0.$$

Absolute error standard:

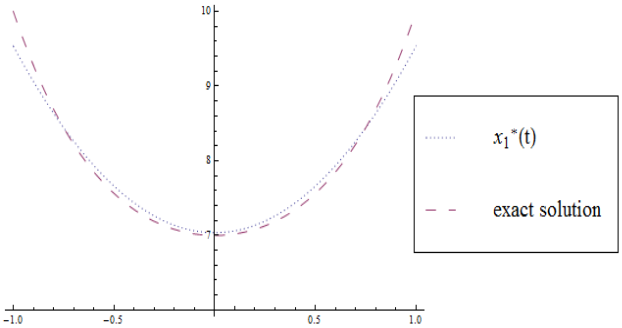

Figure 1. $x_1^*(t)$ graph and the exact solution of the equation (10), $m = 1$.

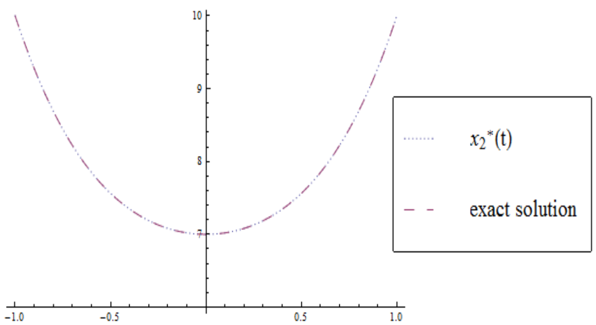

Figure 2. $x_2^*(t)$ graphand exact equation solution (10), $m = 2$.

$$\left\| x^*(t) - x_1^*(t) \right\|_{\tilde{o}} = 0{,}192 \cdot$$

Let $m = 2$. An approximate solution will be the following one:

$$\overline{x_2^*} = \left(x_2^*, \gamma_2^* \right),$$

Where

$$x_2^*(t) = \frac{0{,}6875 \sin 3 \arccos t + 0{,}0625 \sin 5 \arccos t}{\sqrt{1-t^2}} + 7{,}625,$$

$$\gamma_2^* = 0 \,.$$

Absolute error norm:

$$\left\| x^*(t) - x_2^*(t) \right\|_X = 0{,}000 \,.$$

4. Summary

The peculiarity of wavelet analysis is that it may use a large number of basic wavelet functions. Therefore, there is a possibility of choice between the families of wavelet functions and a flexible application of those which solve a particular task most efficiently. Because of the singular integral equation nature, Chebyshev's wavelets of the II-nd type were chosen for the study. A computational scheme of collocation methods was developed by the selected wavelets. The approximation of wavelet functions is checked nearby using computing algebra system Wolfram Mathematic. The operation of the system showed that a good rate of convergence of an approximate equation solution to the exact one was obtained in the second layer.

4.1 Conflict of Interest

The authors acknowledge that the presented data do not contain any conflict of interest.

5. Acknowledgement

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

The authors thank the candidate of physical and mathematical sciences in for her attention to this work and for valuable remarks.

6. References

1. Heidari M. Recurrent fuzzy wavelet neural network to control a PCV. Indian Journal of Science and Technology. 2015 Dec; 8(36):1–6.
2. Gabdulkhaev BG. Direct methods of singular integral equation solutions of the Ist kind. Kazan: Publishing House of Kazan University; 1994. p. 34–65.
3. Erdogan F. Approximate solutions of systems of singular integral equations. SIAM Journal on Applied Mathematics. 2012 Feb; 17(6):1041–59.
4. Astafyeva NM. Wavelet analysis: theory basics and application examples. Physical Science Achievements. 1996 Nov; 166(11):1145–70.
5. Johansson E. Wavelet Theory and some of its Applications. Lulea University of Technology; 2005 Feb. p. 1–90.

6. Sohrabi S. Comparision Chebyshev wavelets method with BPFs method for solving Abel's integral equation. Ain Shams Engineering Journal. 2011 Sep-Dec; 2 (3-4):249–54.
7. Chui CK, Mhaskar HN. On trigonometric wavelets. Constructive Approximation. 1993 Jun; 9(2):167–90.
8. Kilgore T, Prestin J. Polynomial wavelets on an interval. Constructive Approximation. 1996 Mar; 12(1):95–110.
9. Capobianho MR, Themistolakis W. Interpolating polynomial wavelet on [−1, 1]. Advanced Computer Mathematics. 2004 Nov; 23(4):353–74.
10. Wang Y, Fan Q. The second kind Chebyshev wavelet method for solving fractional differential equations. Applied Mathematical Computing. 2012 May; 218(17):8592–601.
11. Celik I. Chebyshev wavelet collocation method for solving generalized burgers-huxley equation. Mathematical Methods in the Applied Sciences. 2016 Feb; 39(3):366–77.
12. Sultanahmedov M. Special wavelets based on Chebyshev polynomials of the second kind and their approximate properties. Izvestiya of Saratov University New Series Mathematics Mechanics Informatics. 2016; 16(1):34–41.
13. Gabdulkhaev BG. Optimal approximations of linear problem solutions. Kazan: Publishing House of Kazan University; 1980.
14. Rastegar S, Babaeian A, Bandarabadi, M, Toopchi Y. Airplane detection and tracking using wavelet features and SVM classifier. 41st Southeastern Symposium on System Theory; 2009 Mar. p. 64–7.
15. Babaeean A, Tashk AB, Bandarabadi M, Rastegar S. Target tracking using wavelet features and RVM classifier. 4th International Conference on Natural Computation; 2008 Oct. p. 569–72.
16. Babaeian A, Rastegar S, Bandarabadi M, Rezaei M. Mean shift-based object tracking with multiple features. 41st Southeastern Symposium on System Theory Thullahoma; 2009 Mar. p. 68–72.
17. Nowruzpour M. Dynamic response for a functionally graded rectangular plate subjected to thermal shock based on LS theory. Applied Mechanics and Materials, Transaction on Technical Publications. 2013 Jul; 332:381–95.
18. Vaziri MR. Modification of shock resistance for cutting tools using functionally graded concept in multilayer coating. Journal of Thermal Science and Engineering Applications. 2015 Mar; 7(1):1–8.
19. Nowruzpour M, Mohsen S, Naei MH. Tow dimensional analysis of functionally graded partial annular disk under radial thermal shock using hybrid fourier-Laplace Transform. Applied Mechanics and Materials, Transaction on Tech Publications. 2013 Oct; 436:92–9.
20. Mehrian SM, Zamani Mehrian S. Modification of space truss vibration using piezoelectric actuator. Applied Mechanics and Materials, Transaction on Technical Publications. 2015 Nov; 811:246–52.
21. Mehrian SM, Nazari A, Naei MH. Coupled thermo-elasticity analysis of annular laminate disk using laplace transform and Galerkin finite element method. Applied Mechanics and Materials, Trans Tech. 2014 Oct; 656:298–304.
22. Mehrian SHZ, Amrei SAZ, Maniat M, Nowruzpour M. Structural health monitoring using optimizing algorithms based on flexibility matrix approach and combination of natural frequencies and mode shapes. International Journal of Structural Engineering. 2016; 7(4):1–8.