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# A Robust Model Predictive Control for Balancing of an Inverted Pendulum

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## Abstract

This paper presents a methodology to design robust predictive controller for the balancing of an inverted pendulum. The inverted pendulum is one of the most difficult control problem in which the pendulum needs to be balanced against the cart, which moves only in two directions to the left or to the right. A new robust controller is designed to balance the pendulum and produces results which are more effective and fast. A Model Predictive Control (MPC) and PID control strategies are applied for controlling the system equations of the inverted pendulum model and are analyzed and compared. The results of controllers implemented in MATLAB shows that both the strategies are able to control the system but robust model predictive control strategy gives better response as compared to conventional PID controller.

**Keywords:** Controller, Inverted Pendulum (IP), Model Predictive Control (MPC), Proportional Integral Derivative (PID)

## 1. Introduction

Model Predictive Control (MPC) refers to a control strategy in which model of the process to be controlled is used to predict the future control inputs and plant responses with optimization at regular sequences[1]. A control algorithm is designed to optimize the future output of the plant based on these predictions[2]. MPC improves the performance and control of many applications in the process industry and has become one of the most widespread control strategies for processes with constraints[3]. MPC has a very well grounded theoretical basis and its algorithms are designed to control systems with many control variables and it deals with the constraints on state and input in a very systematic manner. The predicted inputs are calculated from the design problem by solving it in real time.

Model Predictive Control benefits linear systems which attracted many researchers attention[4–7]. Similarly, nonlinear model predictive control is also applied in different process industries for many years to control complex problems which is the main attraction strategy[8–10].

One of the most classical problems in the field of control engineering is the balancing of inverted pendulum while the cart moves horizontally. IP is one of the most reasonable platforms to test classical and modern control techniques[11,12]. A man's weight exerts force downward while standing on earth against which (exactly opposite) he balances himself. By moving the position of the two legs he manages to balance himself against the force of gravitation. It is a perfect example of the man who balances himself on the earth.

The system stabilization will be achieved through control techniques via PID and Model Predictive Control[13]. A comparative study and analysis between these control techniques will be made in order to suggest an optimal method to control the pendulum[14].

## 2. Model of the Inverted Pendulum System

Inverted pendulum is the single input multi output systems, which is highly unstable and nonlinear. The model of the inverted pendulum is designed and control strategies are applied to stabilize the system efficiently and robustly.

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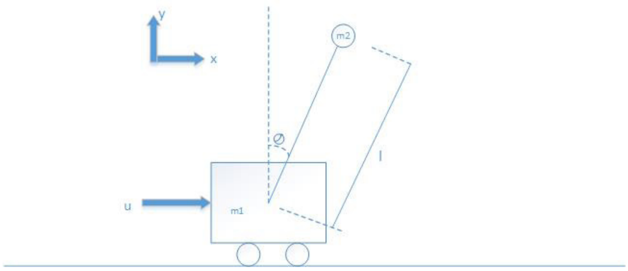



**Figure 1.** Inverted pendulum.

## 2.1 Equations of the System

The horizontal motion of the cart is given by summing all the horizontal forces,

$$F = ma \tag{1}$$

It can be written according to the figure as:

$$u - b\dot{x} = m_1\ddot{x} + m_2\ddot{x}_G \tag{2}$$

Where,

$$x_G = x + l\sin\theta \tag{3}$$

$$\ddot{x}_G = \ddot{x} + l\frac{d^2}{dt^2}(\sin\theta) \tag{4}$$

Put Equation (3) in the above mentioned Equation (2) to get:

$$u = m_1\ddot{x} + m_2\left(\ddot{x} + l\frac{d^2\sin\theta}{dt^2}\right) + b\dot{x} \tag{5}$$

$$\frac{d^2}{dt^2}(\sin\theta) = \frac{d}{dt}\left(\frac{d\theta}{dt}\cos\theta\right) \tag{6}$$

applying derivatives so we get,

$$\frac{d^2\sin\theta}{dt^2} = -\sin\theta(\dot{\theta})^2 + \cos\theta(\ddot{\theta}) \tag{7}$$

Substituting Equation (7) in Equation (5) to get,

$$u = m_1\ddot{x} + m_2\ddot{x} + m_2 l\cos\theta(\ddot{\theta}) - m_2 l\sin\theta(\dot{\theta})^2 + b\dot{x} \tag{8}$$

For the vertical motion of the pendulum the Newton's second law can be written as:

$$u_y - m_2 g = m_2\ddot{y}_G \tag{9}$$

Where,

$$y_G = l\cos\theta \tag{10}$$

and

$$\ddot{y}_G = -l\cos\theta(\dot{\theta})^2 - l\sin\theta(\ddot{\theta}) \tag{11}$$

Substitute Equation (11) in above mentioned Equation (9)

$$u_y - m_2 g = m_2\left[-l\cos\theta(\dot{\theta})^2 - l\sin\theta(\ddot{\theta})\right] \tag{12}$$

$moment = I\ddot{\theta}$ and $torque = Ur$

Both the equations are compared and can be expressed as:

$$u_y I\sin\theta - u_x I\cos\theta = I\ddot{\theta} \tag{13}$$

Substituting the values of horizontal force and vertical force in Equation (13):

$$\left[\left(m_2 g + m_2 - l\cos\theta(\dot{\theta})^2 - l\sin\theta(\ddot{\theta})\right)\right]I\sin\theta - \left[\left(m_2\ddot{x} + m_2 l\cos\theta(\ddot{\theta}) - m_2 l\sin\theta(\dot{\theta})^2 + b\dot{x}\right)\right]I\cos\theta = I\ddot{\theta} \tag{14}$$

The final equation we get is given by:

$$(\mathrm{I} + \mathrm{m_2 l^2})\ddot{\theta} = m_2 gl\sin\theta - m_2 l\cos\theta\ddot{x} \tag{15}$$

These equations are needed to be linearzied about $\theta = \pi$, the pendulum is $\pi$ radians from its position where it should be stabilized.

$$\theta = \pi + \varphi \tag{16}$$

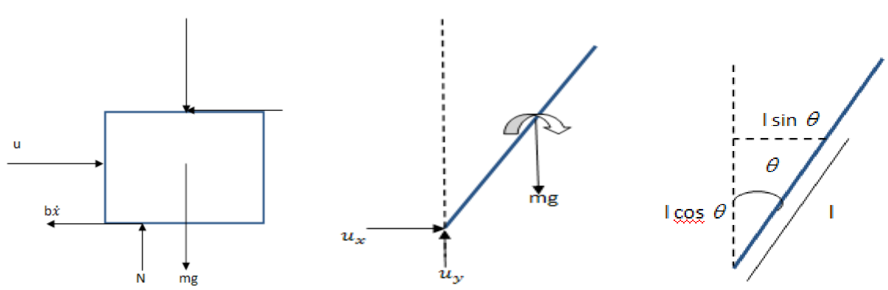



**Figure 2.** Cart and pendulum free body diagram.

$\varphi$ is the small angle of deflection which should approach to zero for the pendulum to be stable ($\varphi \to 0$), $\sin\varphi = \varphi$, $\cos\varphi = 1$. The derivative of $\varphi$ is zero because the deflection is so small (Assume $\dot{\varphi}^2$ as zero) and $\sin\theta = -\varphi$, $\cos\theta = -1$. Putting all the required values so we can get the final equations as expressed below:

$$u = (m_1 + m_2)\ddot{x} - m_2 l(\ddot{\varphi}) + b\dot{x} \tag{17}$$

$$(I + m_2 l^2)\ddot{\varphi} = m_2 l\ddot{x} - m_2 gl\varphi \tag{18}$$

If we made an assumption under which the centre of the mass pendulum is equal to its gravity centre, then I = 0. The equation becomes as:

$$l\ddot{\varphi} = \ddot{x} - g\varphi \tag{19}$$

Now, taking Laplace transform of both Equations (17) and (18) respectively.

$$U(s) = (m_1 + m_2)s^2 X(s) - m_2 ls^2\varphi(s) + bsX(s) \tag{20}$$

$$ls^2\varphi(s) = s^2 X(s) - g\varphi(s) \tag{21}$$

Solving Equation (21) for X(s), we get,

$$X(s) = [(ls^2 + g)/s^2]\varphi(s) \tag{22}$$

Put the specified equation in (20) to get,

$$U(s) = (m_1 + m_2)s^2[(ls^2 + g)\varphi(s)/s^2] - m_2 ls^2\varphi(s) + bs[(ls^2 + g)\varphi(s)/s^2] \tag{23}$$

The transfer function of the inverted pendulum system is given as:

$$\frac{\varphi(s)}{U(s)} = \frac{s}{(m_1 + m_2)[(ls^2 + g)]s - m_2 ls^3 + b[(ls^2 + g)]} \tag{24}$$

For friction to be zero i.e. b = 0,

$$\frac{\varphi(s)}{U(s)} = \frac{1}{(m_1 + m_2)[(ls^2 + g)] - m_2 ls^2} \tag{25}$$

# 3. Model Predictive Control Strategy

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \tag{26}$$

Where, $u_k$ is the input and $x_k$ is state vector at the $k^{th}$ sampling time sequence. A succession of state predictions are created, when the model is simulated over N sampling sequences. The prediction sequence is stacked into vectors u, x.

$$u_k = \begin{matrix} u_{k|j} \\ u_{(k+1|j)} \\ u_{(k+2|j)} \\ \vdots \\ u_{(k+N-1|j)} \end{matrix} \tag{27}$$

$$x_k = \begin{matrix} u_{(k+1|j)} \\ u_{(k+2|j)} \\ u_{(k+3|j)} \\ \vdots \\ u_{(k+N|j)} \end{matrix} \tag{28}$$

Where, N = 1,2,3…. and $u_{(k+i|j)}$ and $x_{(k+i|j)}$ denotes the vectors which is well expressed as input and state at the given time interval k + i and the predictions are made at the time sequence j.

$$\mathbf{x}_{(k+i+1|j)} = \mathbf{A}\mathbf{x}_{(k+i|j)} + \mathbf{B}\mathbf{u}_{(k+i|j)}, \quad i=0,1,2\ldots. \tag{29}$$

State vector at the beginning is expressed as: $\mathbf{x}_{(k|j)} = \mathbf{x}_{(k)}$.

## 3.1 Prediction Models and Optimization

The predicted cost is given by:

$$J(k) = \sum_{i=0}^{N}\left[x^T_{(k+i|j)}Qx_{(k+i|j)} + u^T_{(k+i|j)}Ru_{(k+i|j)}\right] \tag{30}$$

Q is ( positive definite and positive semi-definite), where R is only (positive definite) matrix. The predicted cost $J(k)$ is the function of input vector $u_k$ and input to the design problem for minimizing the cost is given by $u^*_k$.

$$u^*_k = arg\,\underset{u}{MINIMUM}\,J(k) \tag{31}$$

Input to the plant is the only initial element of the input sequence $u^*_k$, which is predicted first is $u_k = u^*_{(k|j)}$. The process in which the predicted cost is minimized, by computing $u^*_k$ and then implement the initial element of predicted input u*. This is a cyclic process repeated at each interval k= 0, 1, 2, 3…. This is known as online optimization.

## 3.2 Prediction Equations

The prediction cost function depends on input sequence u$u_k$ $u_k$. So, the predicted quadratic cost can be showed as a function of input u.

$$J(k) = u_k^T Hu(k) + 2f^T u_k + g \tag{32}$$

H is defined as positive definite matrix and can also be positive semi-definite matrix. F and g are the vector and scalar quantities respectively depends on x (k).

$$A_c u(k) \le b_c \tag{33}$$

Where $A_c$ is a constant matrix and $b_c$ is a vector which is a function of x (k).

$$\underset{u}{MINIMIZE}\ u^T Hu + 2f^T u \quad (34)\ (35)$$
$$subject \rightarrow A_c u \le b_c$$

This type of optimization is known as quadratic programming, where positive definite matrix is H. So, the equations comprise of function and constraints are linear, which is convex functions of the optimization input variable u. Shows that it is a convex problem.

If a prediction model that is nonlinear developed for the process to be controlled than the optimization of the design problem is complex as compared to linear model. Due to the nonlinear dependencies of the state $x_k$ on input vector $u_k$. The predicted cost function can be expressed as $J(u_k, x_k)$ and the constraints as $g(u_k, x_k) \le 0$, which is the non-convex functions of input $u_k$. So, the design optimization problem becomes a non-convex nonlinear programming. Given as:

$$\underset{u}{MINIMIZE}\ J(u, x_k) \quad (36)$$

$$subject \rightarrow g(u, x_k) \le 0 \quad (37)$$

## 4. PID Controller

The inverted pendulum model is implemented and the results of the simulation displays that the system model is highly unstable in open-loop, as it is evident from the pole-zero map of the system that one of the poles lies strictly inside the unstable region. The step and impulse response in the (Figure 3) shows instability in the system due to very speedily divergence of the theta. This characteristics of the model gives us a response which specify instability in the system. The stability plot of the inverted pendulum system reveals highly unstable nature of the system as the branch of locus recline in the unstable side (left hand side) of the imaginary axis due to which for all gain values the system is also unstable in closed loop. The locus needs to be reshaped for the system so that all the poles of the system will lie in the stable region (left half plane). The simulation results of the open-loop unstable system is shown in the Figure 3 below.

The root locus is shaped by introducing pole of the controller to cancel the effect of zeros here available at the origin of the locus diagram and those all should be in the region which is stable. Zeros of the compensator are initiated on the left of the s-plane in stable region to tug the locus more in the direction of the stable region as shown in the Figure 4. Hence, the PID controller is implemented and it will stabilize the system to let fall all the roots in the stable region (left s-plane). The poles and zeros of the closed loop system shown in the Figure 4 lies in the stable region (left hand side) and fulfilling the stability criteria. The closed loop system is very stable and the response of the system is shown in the Figure 4 below:

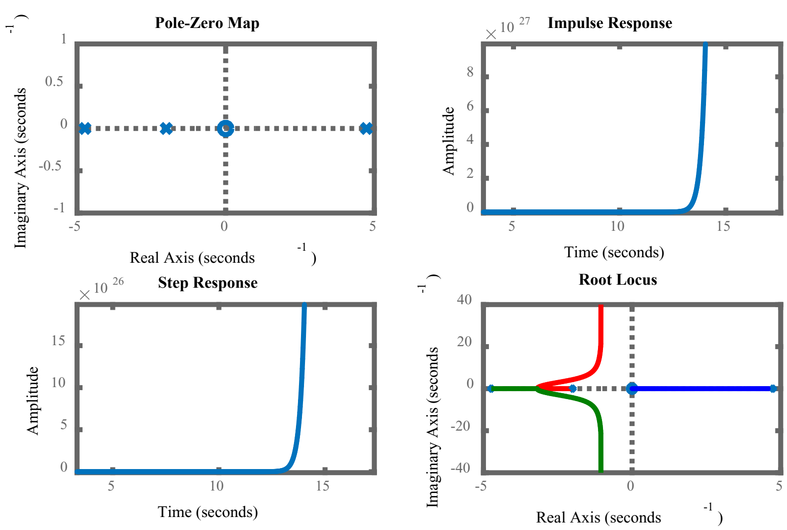



**Figure 3.** Unstable open-loop inverted pendulum system.

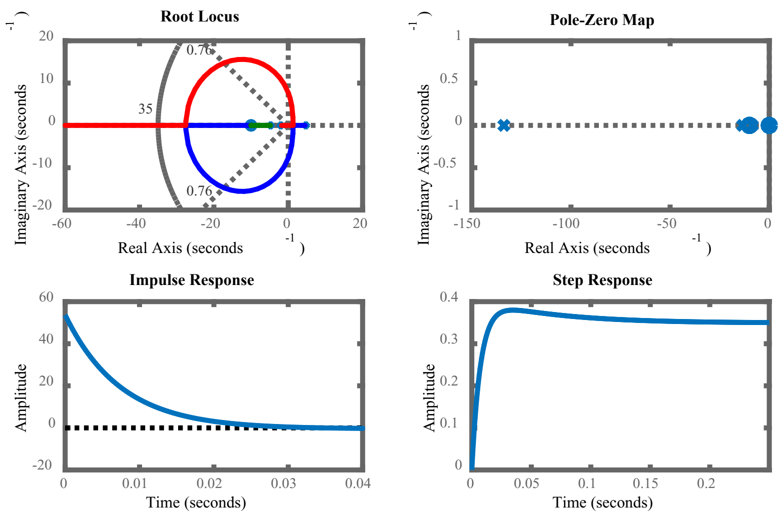
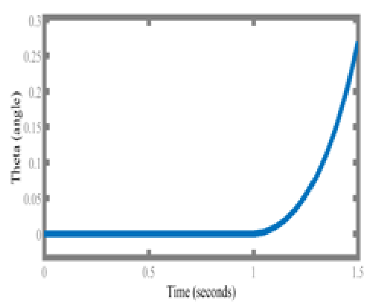


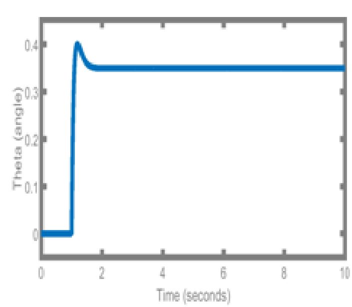

**Figure 4.** Stable closed-loop inverted pendulum system.

## 5. Simulation and Results

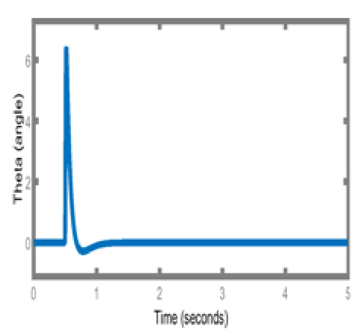
Simulation results of the inverted pendulum system designed and when theta diverges very quickly it displays that the system is very unstable. The system becomes stable as theta for the model of the system is stabilized for a certain value, it is very much apparent from the Figure 5. The impulse response of the closed loop inverted pendulum system is stable as it again comes to zero and by introducing disturbance in the force, all the disturbances that are in inverted pendulum model are all rejected and system explicitly gives stable response as shown in the Figure 5 below:



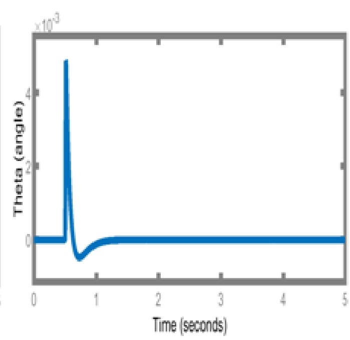

(a) Response of the open loop system.




(b) Response of the closed loop system.




(c) Impulse response of the stable system.




(d) Response to disturbance.

**Figure 5.** Response of open-loop and closed-loop PID system.

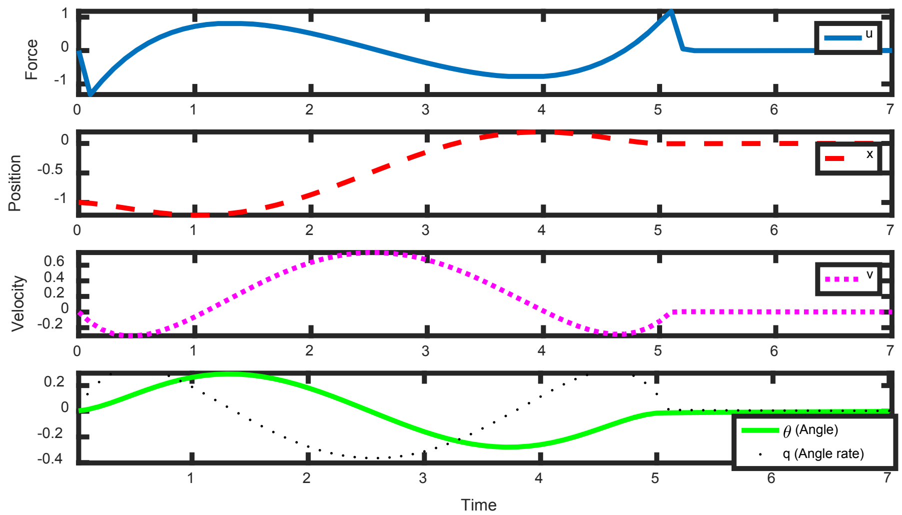



**Figure 6.** Response of model predictive control.

Model predictive control is implemented for the inverted pendulum system (v, theta and q) are zero at the beginning. The result shows that when the force is negative the cart moves backward and the angle starts tipping forward. When force is applied forward (positive) the cart moves in the forward direction the pendulum moves backward.

The pendulum makes the maneuver till 5 seconds and then the cart velocity becomes zero. So, it takes (0.1 seconds) for the system to stabilize itself and gives theta and angle rate zero. Model predictive control gives better response and performance as the results are compared. The steady state error is zero and settling time, rise time is less from PID controller.

## 6. Conclusion

In this paper, a robust predictive controller is successfully designed and implemented in MATLAB for the inverted pendulum system. The controller developed is more fast and produces results which are highly effective as compared to other control strategies. The result reveals that both the control methods are successful in controlling the cart's position and the angle of inverted pendulum system. Based on the simulation results, robust MPC gives better performance in controlling and stabilizing of the inverted pendulum and is more suitable method for controlling of similar unstable systems.

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