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Dynkin Diagrams and Root Systems of Indefinite Quasi-Hyperbolic Kac-Moody Algebra $QHA_4^{(2)}$

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Abstract

Objectives: To obtain the complete classification of a particular class of indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_4^{(2)}$ and to study the properties of imaginary roots. **Methods:** Pure theoretical approach for the classification of Dynkin diagrams and an analytical approach for the root system are applied. **Findings:** The complete classification of Dynkin diagrams associated to the Generalized Cartan Matrix of quasi hyperbolic indefinite type of Kac-Moody algebra $QHA_4^{(2)}$ is obtained. Here, the number of connected, non isomorphic Dynkin diagrams associated with $QHA_4^{(2)}$ is 858. The properties of strictly imaginary and purely imaginary roots are also studied for $QHA_4^{(2)}$. **Applications:** Kac-Moody algebra has applications in various fields of mathematics and mathematical physics such as combinatorics, number theory, partial differential equations, quantum physics etc.

Keywords: Dynkin Diagram, Indefinite Quasi Hyperbolic, Kac-Moody Algebras, Purely Imaginary Roots, Strictly Imaginary Roots

1. Introduction

The theory of Kac-Moody Lie algebras was first initiated and developed by[1,2] around 1968. Since then the subject has been attracting many mathematical researchers because of its significant applications to various branches of mathematics, quantum physics, mathematical physics etc.

The complete classification of Kac-Moody algebras, possessing strictly imaginary property was obtained in Casperson[3]. Kang[4–7] had studied about $HA_1^{(1)}$, $HA_2^{(2)}$ and $HA_n^{(1)}$ using the techniques of [8]. The purely imaginary roots, a new class of Extended-hyperbolic Kac-Moody algebras was introduced by[9] and also the root multiplicities for the particular classes of extended-hyperbolic Kac-Moody algebras $EHA_1^{(1)}$, $EHA_2^{(2)}$ were studied in[10–13].

Another, new class of indefinite non-hyperbolic type of Kac-Moody algebra called Quasi-Hyperbolic Kac-Moody algebra was introduced by[14]. Some particular types of indefinite, non-hyperbolic Kac-Moody algebras QHG_2, $QHA_2^{(1)}$, $QHA_4^{(2)}$, $QHA_5^{(2)}$ and $QHA_7^{(2)}$ were realized as graded Kac-Moody algebras of quasi hyperbolic type and also the homology modules up to level three and the structure of the components of the maximal ideals up to level four were computed by [15–19]. The complete classifications of the Dynkin diagrams and some properties of real and imaginary roots for the associated Quasi affine Kac Moody algebras $QAC_2^{(1)}$ were obtained by[20],[21] obtained the complete classification of the Dynkin diagrams associated to the indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_2^{(1)}$ and also studied the properties of purely and strictly imaginary roots.

In this work, we consider the particular class of Quasi-Hyperbolic type of Kac-Moody algebra $QHA_4^{(2)}$ whose associated symmetrizable and indecomposable GCM is

$\begin{pmatrix} 2 & -1 & 0 & -p \\ -2 & 2 & -1 & -q \\ 0 & -2 & 2 & -r \\ -l & -m & -n & 2 \end{pmatrix}$ where p,q,r,l,m,n are non-negative

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integers. The main aim of this paper is to give a complete classification of the Dynkin diagrams associated with $QHA_4^{(2)}$ and study the properties of strictly imaginary roots and purely imaginary roots for the same.

Preliminaries

In this section, we recall some basic concepts of Kac-Moody algebras[1–3].

1.1.1 Definition 1.1[1]:

A realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ of rank l, is a triple (H, π, π^{v}), H is a 2n - l dimensional complex vector space, $\pi = \{\alpha_1,\dots,\alpha_n\}$ and $\pi^{v} = \{\alpha_1^{v},\dots,\alpha_n^{v}\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{v}) = a_{ij}$ for i, j = 1,…,n. π is called the root basis. Elements of π are called simple roots. The root lattice generated by π is

$$Q=\sum_{i=1}^{n} z\alpha_i .$$

1.1.2 Definition 1.2[1]

The Kac-Moody algebra g (A) associated with a Generalized Cartan Matrix (abbreviated as GCM) $A=(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i , i= 1,2,...,n and H with the following defining relations:

$$\begin{aligned}
&[h,h'] = 0, \quad h,h' \in H\\
&[e_i,f_j] = \delta_j \alpha_i^{v}\\
&[h,e_j] = \alpha_j(h)e_j\\
&[h,f_j] = -\alpha_j(h)f_j\ ,\ i,j\in N\\
&(ade_i)^{1-a_{ij}} e_j = 0\\
&(adf_i)^{1-a_{ij}} f_j = 0\ ,\forall\ i\neq j,\ i,j\in N
\end{aligned}$$

The Kac-Moody algebra g (A) has the root space decomposition $g(A) = \underset{\alpha\in Q}{\oplus} g_\alpha(A)$ where

$$g_\alpha(A)=\{x\in g(A)/[h,x]=\alpha(h)x,\ for\ all\ h\in H\}$$

An element α, α ≠ 0 in Q is called a root if $g_\alpha \neq 0$. Let

$$Q=\sum_{i-1}^{n} z_+\alpha_i .$$

1.1.3 Definition 1.3 [14]

Let $A=(a_{ij})_{i,j=1}^{n}$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S (A) to be Quasi Hyperbolic (QH) type if S (A) has a proper connected sub diagram of hyperbolic type with (n-1) vertices. The GCM A is of QH type if S (A) is of QH type. We then say that the Kac-Moody algebra g (A) is of QH type.

2. Classifications Of Quasi-Hyperbolic Kac-Moody Algebra $QHA_4^{(2)}$

In this section, a complete classification of Dynkin diagrams is given. Then the properties of roots are studied.

For $QHA_4^{(2)}$, the associated symmetrizable and indecomposable GCM is:

$$A=\begin{pmatrix} 2 & -1 & 0 & -p\\ -2 & 2 & -1 & -q\\ 0 & -2 & 2 & -r\\ -l & -m & -n & 2\end{pmatrix}$$

where p,q,r,l,m,n are non negative integers. The decomposition for this A is given by

A = DB where

$$D=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 0 & l/p\end{pmatrix}\ and\ B=\begin{pmatrix} 2 & -2 & 0 & -p\\ -1 & 1 & -1/2 & -q/2\\ 0 & -1/2 & 1/2 & -r/4\\ -p & -pm/l & -pn/l & 2p/l\end{pmatrix}$$

with the conditions, m = lq/2p, n = lr/4p.

2.1 Theorem 2.1 (Classification Theorem)

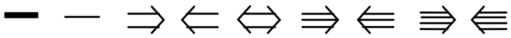
For the Quasi-Hyperbolic indefinite type of Kac-Moody algebra $QHA_4^{(2)}$, whose associated symmetrizable and indecomposable GCM is

$$A=\begin{pmatrix} 2 & -1 & 0 & -p\\ -2 & 2 & -1 & -q\\ 0 & -2 & 2 & -r\\ -l & -m & -n & 2\end{pmatrix}$$

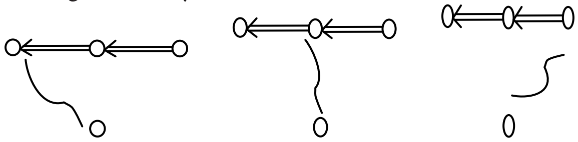
where p,q,r,l,m,n are non negative integers, the number of connected, non isomorphic Dynkin diagrams corresponding to these GCM of $QHA_4^{(2)}$ is 858.

2.1.1 Proof

We consider the Dynkin diagrams obtained from the Dynkin diagram of affine Kac-Moody algebra $A_4^{(2)}$ by adding a fourth vertex, which is generally represented by ,where can be one of the 9 possible edges:



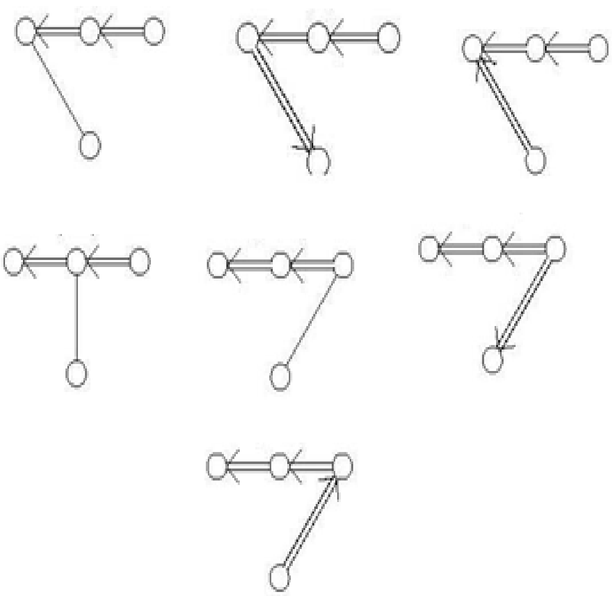
Case 1: In this case, a single edge is added from the fourth vertex to any one of the three vertices of the Dynkin diagram of affine $A_4^{(2)}$. It can be done in the following three ways:



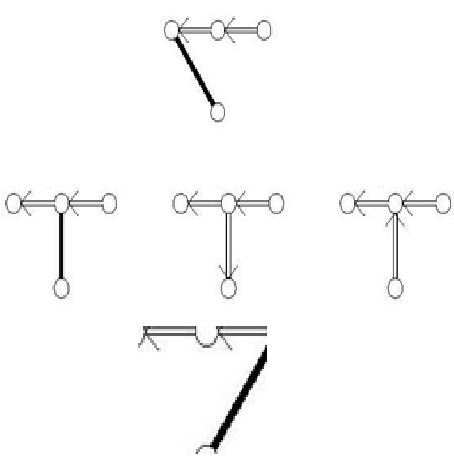
The number of Dynkin diagrams connecting the fourth vertex with any one of the other three vertices is 27.

We discuss the Dynkin diagrams case by case.

1. The following seven Dynkin diagrams are hyperbolic type:

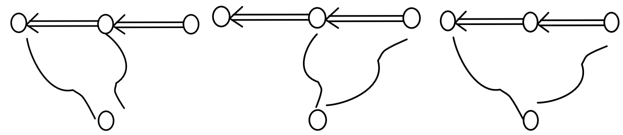


2. The following five Dynkin diagrams do not belong to Quasi-Hyperbolic type:



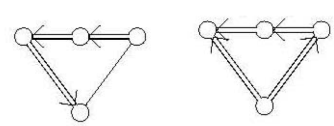
Except these 12 Dynkin diagrams from the total of 27, we get 15 connected, non isomorphic Dynkin diagrams in $QHA_4^{(2)}$.

Case 2: In this case, two edges are added from the fourth vertex to the Dynkin diagram of affine Kac-Moody algebra $A_4^{(2)}$. It can be done in the following three ways:



The number of Dynkin diagrams connecting the fourth vertex with any two of the other three vertices by 9 possible edges as mentioned above is 243.

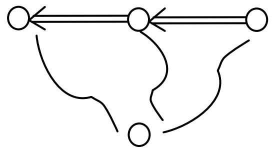
- We discuss the Dynkin diagrams case by case.
- There are seven Dynkin diagrams of hyperbolic type of rank 4, already given in Wan[22].
- The Dynkin diagrams corresponding to the bold face joining the vertices 4 and 2 do not belong to quasi hyperbolic type.
- The Dynkin diagrams corresponding to the bold face joining the vertices 4 and 3 and the vertices 4 and 2 joined by any one of , , do not belong to quasi hyperbolic type.
- The Dynkin diagrams corresponding to the bold face joining the vertices 4 and 1 and the vertices 4 and 2 joined by any one of , , do not belong to quasi hyperbolic type.
- The Dynkin diagrams corresponding to the bold face joining the vertices 4 and 1 and the vertices 4 and 3 joined by any one of bold face, , , do not belong to quasi hyperbolic type.
- The Dynkin diagrams corresponding to the bold face joining the vertices 4 and 3 and the vertices 4 and 1 joined by any one of , , do not belong to quasi hyperbolic type.
- Apart from these, there are two more Dynkin diagrams which do not belong to quasihyperbolic type:



Thus, there are 33 from Dynkin diagrams that do not belong to Quasi-Hyperbolic type.

Except, these 7 + 33 = 40 Dynkin diagrams from the total of 243, we get 203 connected, non isomorphic Dynkin diagrams in $QHA_4^{(2)}$.

Case 3: In this case, three edges are added from the fourth vertex.



The number of Dynkin diagrams connecting the fourth vertex with all the other three vertices by 9 possible edges as mentioned above is 729.

- We discuss the Dynkin diagrams case by case.
- The number of Dynkin diagrams corresponding to the bold face connecting the vertices 4 and 2 which do not belong to quasi hyperbolic types is 9 x 9 = 81.
- The number of Dynkin diagrams corresponding to the bold face connecting the vertices 4 and 1, 4 and 3 which do not belong to quasi hyperbolic type is 8.

Except, these 89 diagrams which are not Quasi hyperbolic type from the total of 729 we get 640 connected, non isomorphic Dynkin diagrams in $QHA_4^{(2)}$.

Therefore, from the above three cases we get, 15 + 203 + 640 = 858 connected, non isomorphic Dynkin diagrams of indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_4^{(2)}$.

These 858 Dynkin diagrams are given in Figure 1.

2.2 Properties of Imaginary Roots

2.2.1 Proposition 2.2

Consider the indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_4^{(2)}$, whose associated symmetrizable and indecomposable GCM is

$$A=\begin{pmatrix} 2 & -1 & 0 & -p \\ -2 & 2 & -1 & -q \\ 0 & -2 & 2 & -r \\ -l & -m & -n & 2 \end{pmatrix}$$

where p,q,r,*l*,m,n are non negative integers. Then the Kac-Moody

algebra g (A) corresponding to $QHA_4^{(2)}$ has the following properties:

- The imaginary roots of g (A) satisfy the purely imaginary property. (i.e. Two imaginary roots add up to another imaginary root).
- The imaginary roots of g (A) satisfy the strictly imaginary property.

2.2.2 Proof

- Since A is a connected, symmetrizable and indecomposable GCM, by using corollary 3.11 in [9], we get, $\Delta_+^{pim}(A)=\Delta_+^{im}(A)$. Hence g (A) has purely imaginary property.
- Since A is symmetrizable and indecomposable GCM, A satisfies the required condition for the Theorem (23) in Casperson[3]. Hence g (A) has strictly imaginary property.

Example 1: Consider the indefinite Quasi-Hyperbolic Kac-Moody algebra $QHA_4^{(2)}$ whose associated symmetrizable and indecomposable GCM.

$$A=\begin{pmatrix} 2 & -1 & 0 & -2 \\ -2 & 2 & -1 & -2 \\ 0 & -2 & 2 & 0 \\ -2 & -2 & 0 & 2 \end{pmatrix}.$$

Since A is symmetrizable, A can be expressed as A = DB where

$$D=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} 2 & -1 & 0 & -2 \\ -1 & 1 & -1/2 & -1 \\ 0 & -1/2 & 1/2 & 0 \\ -2 & -1 & 0 & 2 \end{pmatrix}.$$

Here $(\alpha_1,\alpha_1) = 2$, $(\alpha_2,\alpha_2) = 1$, $(\alpha_3,\alpha_3) = 1/2$, $(\alpha_4,\alpha_4) = 2$, $(\alpha_1,\alpha_2) = (\alpha_2,\alpha_1) = -1$, $(\alpha_1,\alpha_3) = (\alpha_3,\alpha_1) = 0$, $(\alpha_1,\alpha_4) = (\alpha_4,\alpha_1) = -2$, $(\alpha_2,\alpha_3) = (\alpha_3,\alpha_2) = -1/2$, $(\alpha_2,\alpha_4) = (\alpha_4,\alpha_2) = -1$, $(\alpha_3,\alpha_4) = (\alpha_4,\alpha_3) = 0$.

Let $\alpha = \alpha_1+\alpha_2+\alpha_3+\alpha_4$, then $(\alpha, \alpha)= -7/4 < 0$. Therefore α is an imaginary root. For every real root β, we find that $\alpha + \beta$ is also a root. Hence α is a strictly imaginary root. Let $\gamma = \alpha_1+ \alpha_4$, then $(\gamma , \gamma) = 0$. Therefore γ is an imaginary root. We can see that $\alpha +\gamma$ is an imaginary root. Hence α is also purely imaginary.

Example 2: Consider the indefinite Quasi-Hyperbolic Kac-Moody algebra $QHA_4^{(2)}$ whose associated symmetrizable and indecomposable GCM.

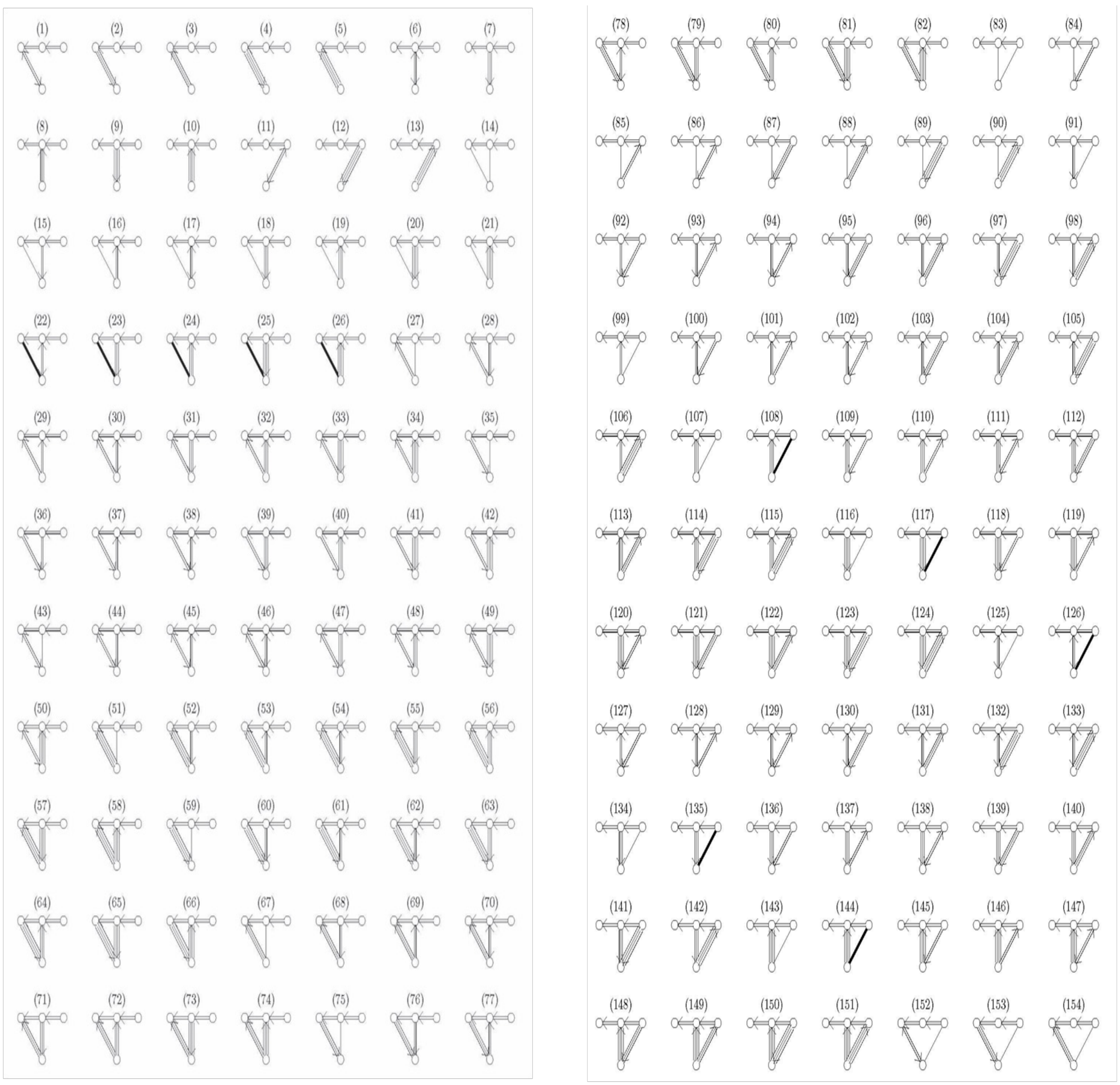
$$A=\begin{pmatrix} 2 & -1 & 0 & -2 \\ -2 & 2 & -1 & -2 \\ 0 & -2 & 2 & -2 \\ -2 & -2 & -2 & 2 \end{pmatrix}$$

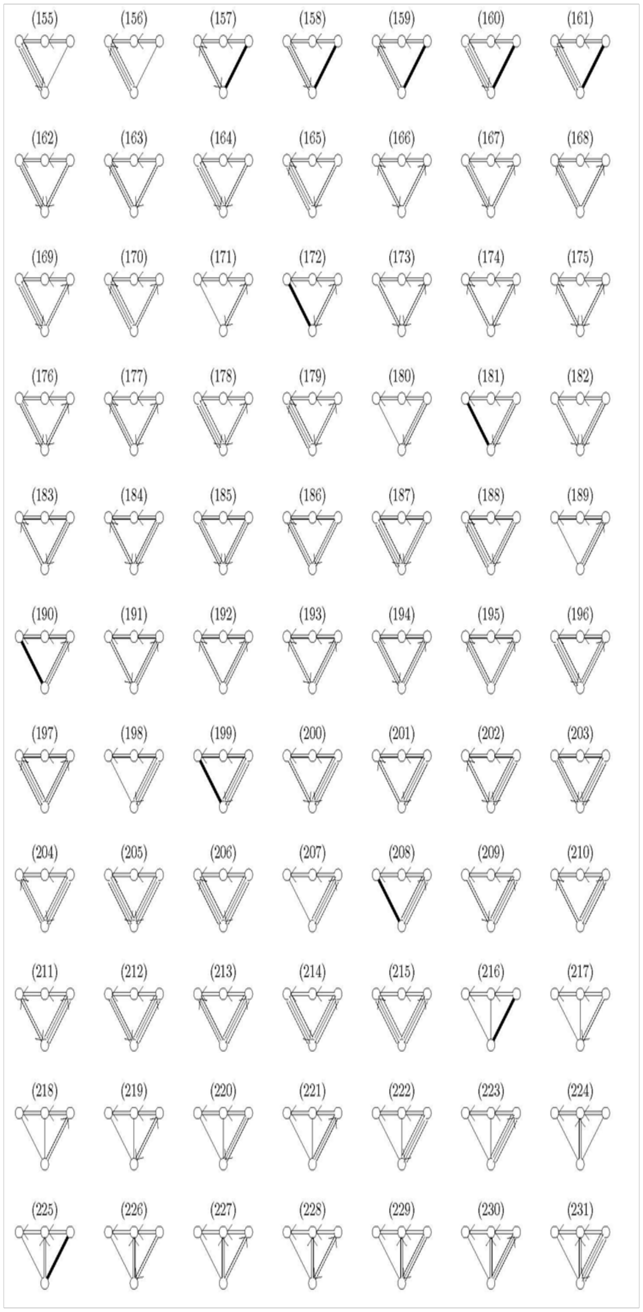
. A can be decomposed into DB where $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 0 & -2 \\ -1 & 1 & -1/2 & -1 \\ 0 & -1/2 & 1/2 & -1/2 \\ -2 & -1 & -1/2 & 2 \end{pmatrix}$.

Here, $(\alpha_1,\alpha_1) = 2$, $(\alpha_2,\alpha_2) = 1$, $(\alpha_3,\alpha_3) = 1/2$, $(\alpha_4,\alpha_4) = 2$, $(\alpha_1,\alpha_2) = (\alpha_2,\alpha_1) = -1$, $(\alpha_1,\alpha_3) = (\alpha_3,\alpha_1) = 0$, $(\alpha_1,\alpha_4) = (\alpha_4,\alpha_1) = -2$, $(\alpha_2,\alpha_3) = (\alpha_3,\alpha_2) = -1/2$, $(\alpha_2,\alpha_4) = (\alpha_4,\alpha_2) = -1$, $(\alpha_3,\alpha_4) = (\alpha_4,\alpha_3) = -1/2$.

Let $\alpha = \alpha_1+\alpha_2+\alpha_3+\alpha_4$, then $(\alpha, \alpha) = -9/2 <0$. Therefore, α is an imaginary root. For every real root γ, we can see that $\alpha + \gamma$ is also a root. Hence α is a strictly imaginary root. Let $\beta = \alpha_2+\alpha_3+\alpha_4$ then $(\beta,\beta) = -1/2 <0$Therefore, β is also an imaginary root. Here, we find that, $\alpha+\beta$ is also an imaginary root. Hence α is also purely imaginary.

Figure 1. Connected, non isomorphic Dynkin diagrams of Quasi hyperbolic Kac-Moody algebra $QHA_4^{(2)}$.

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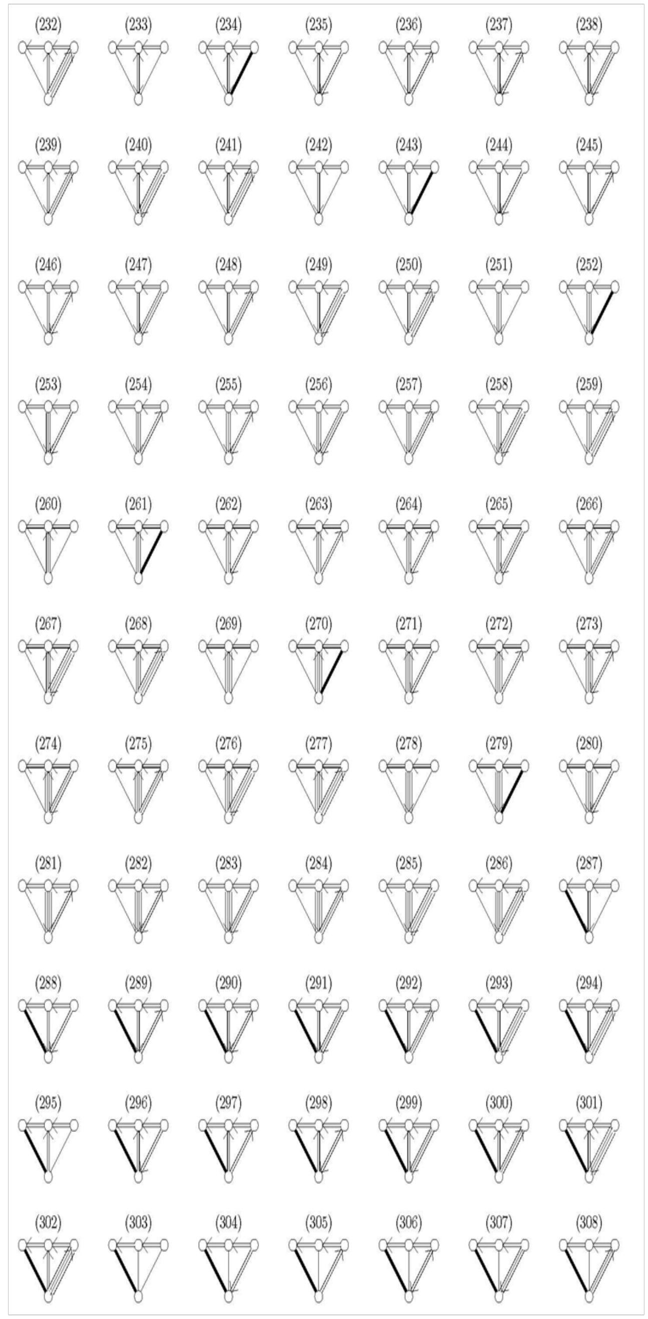
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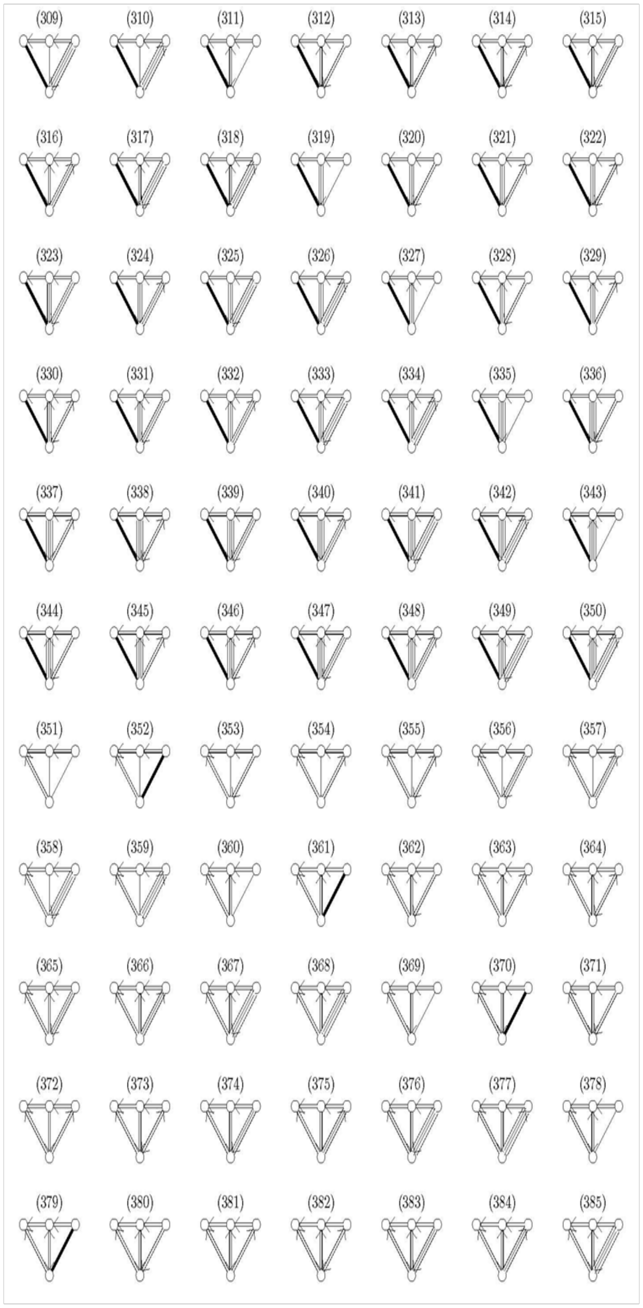
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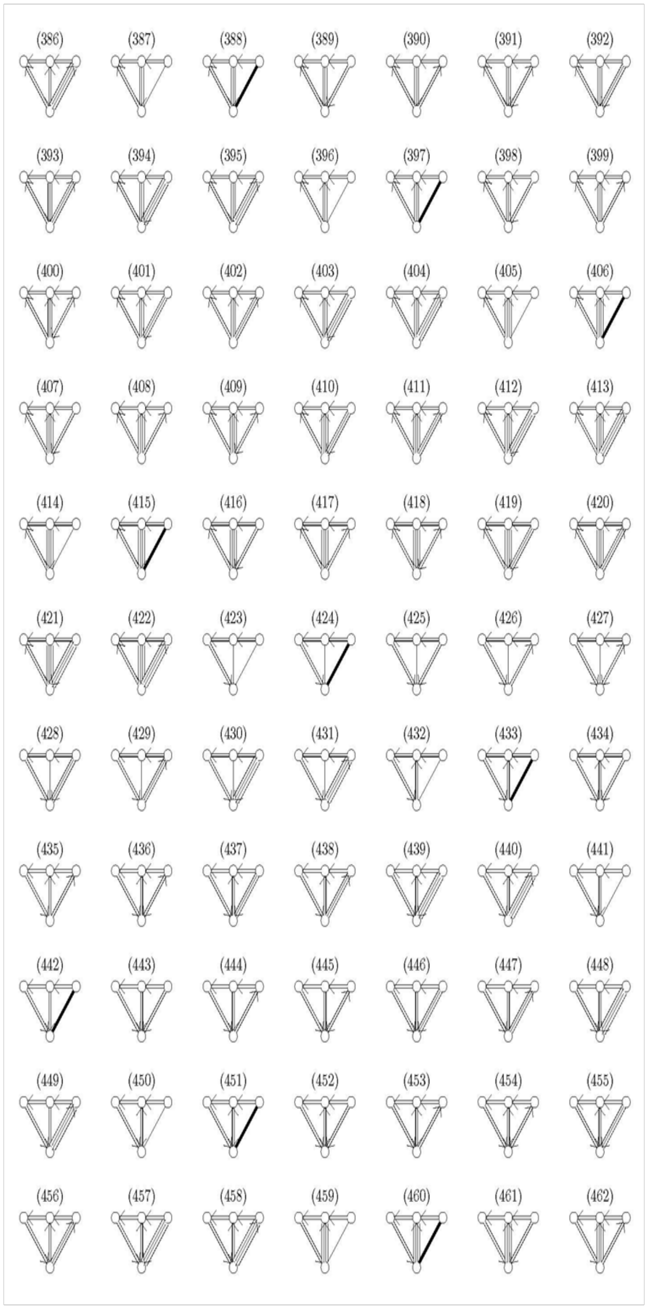
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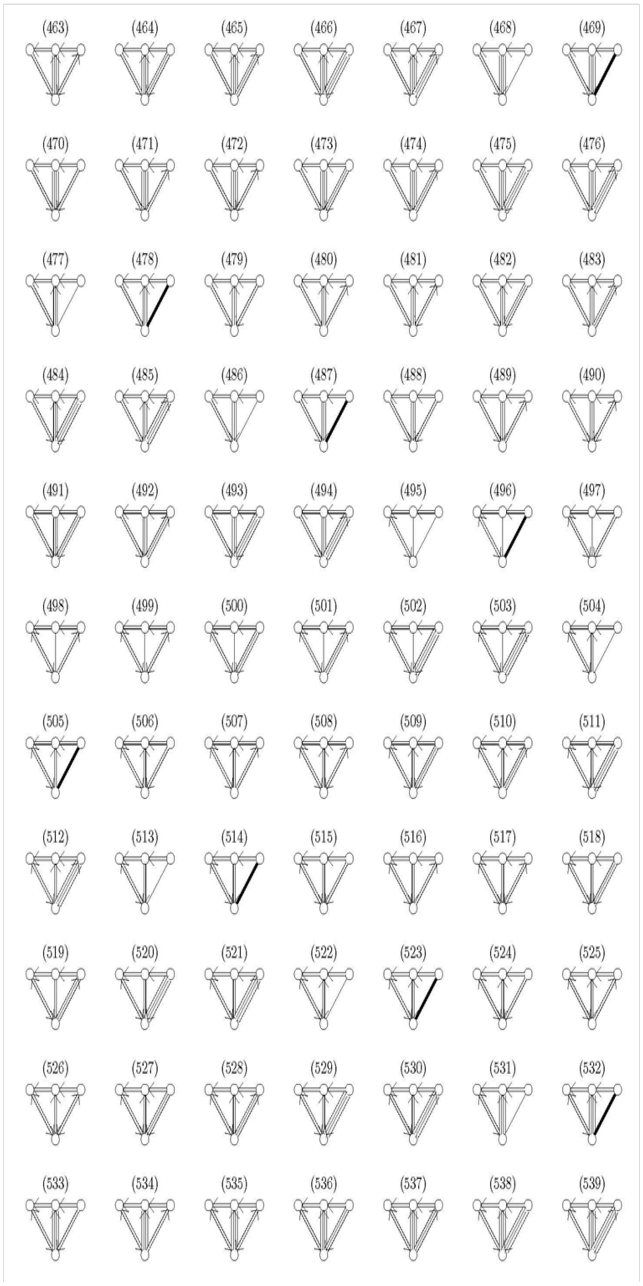
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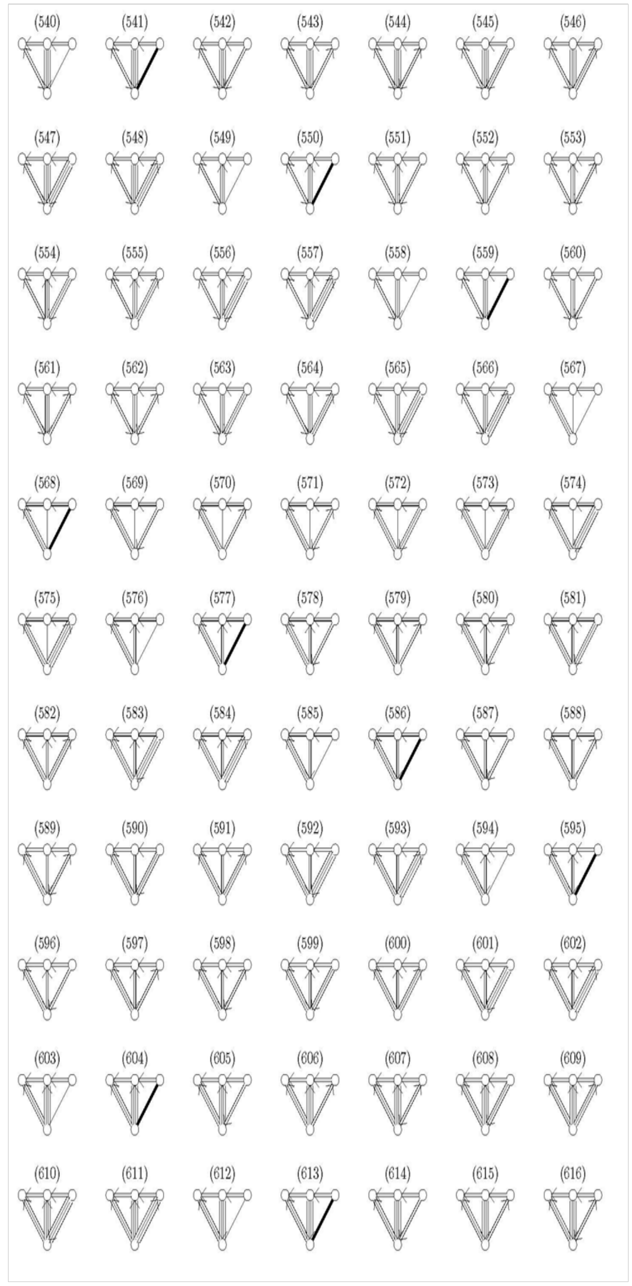
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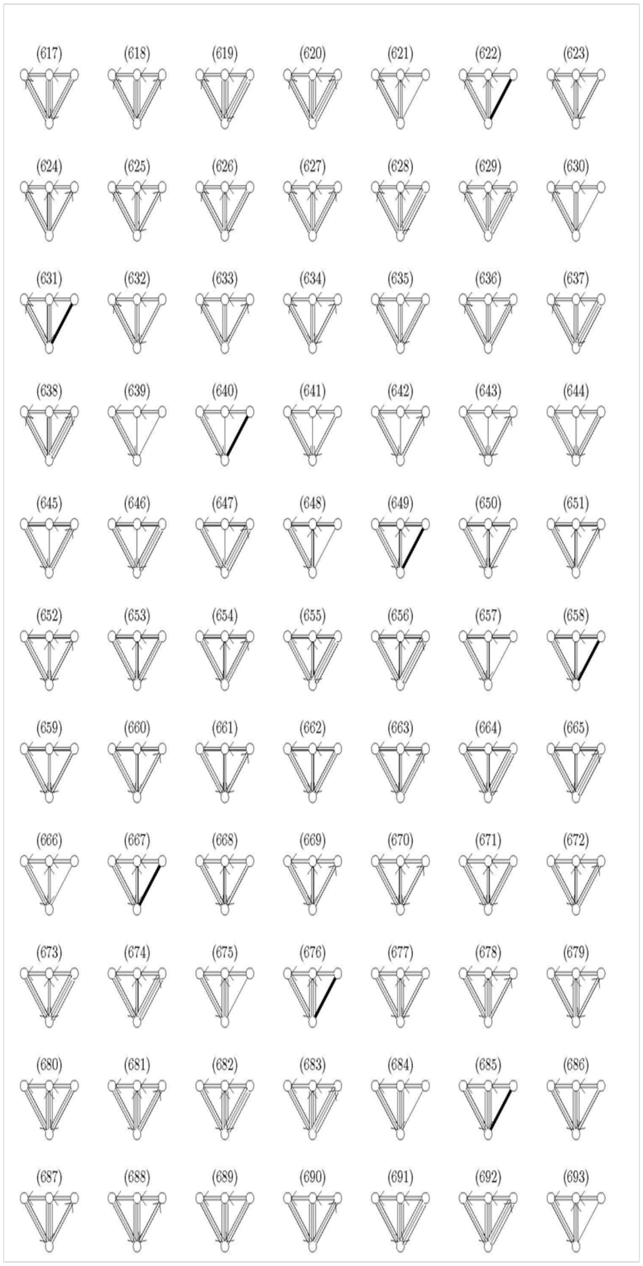
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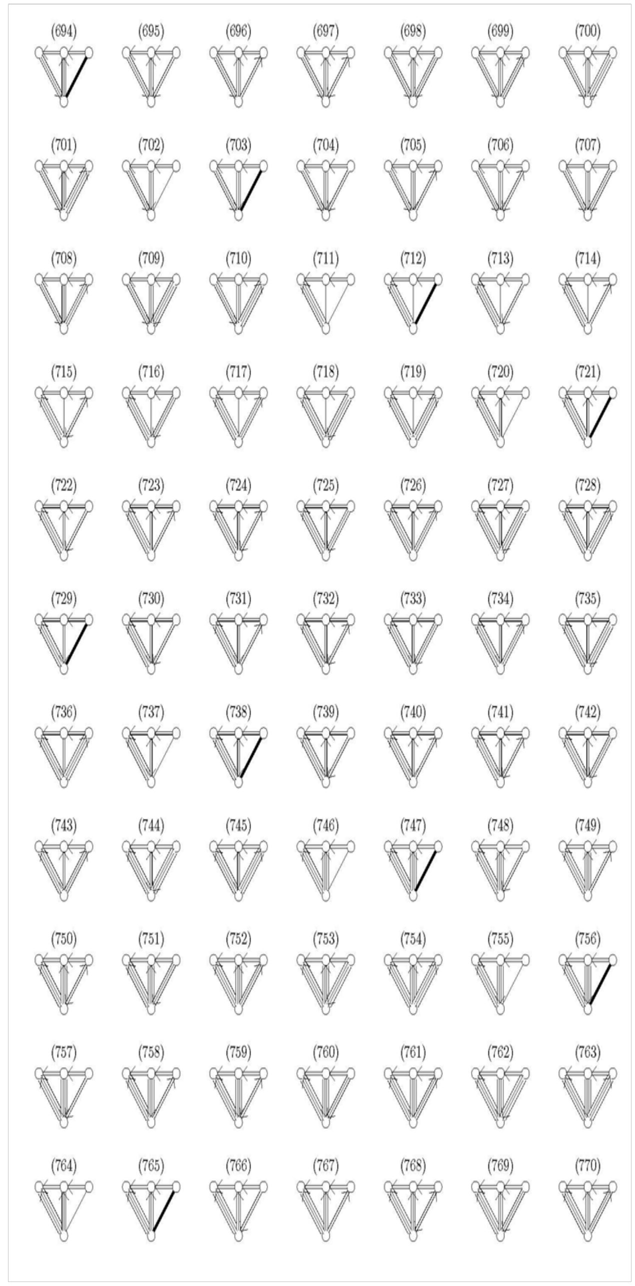
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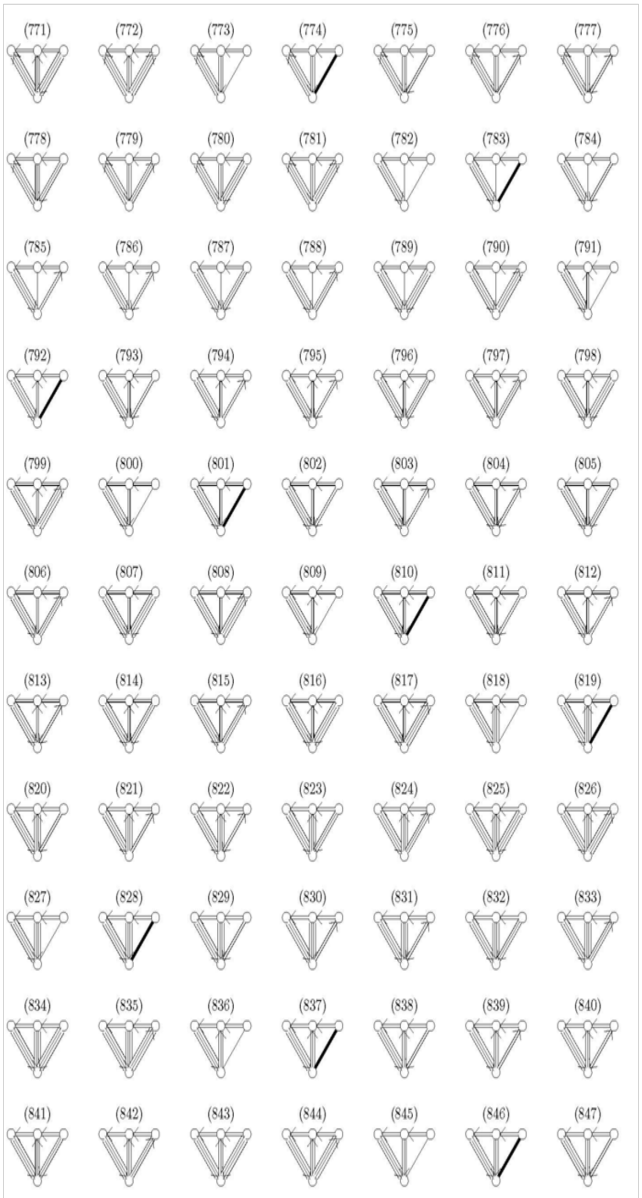

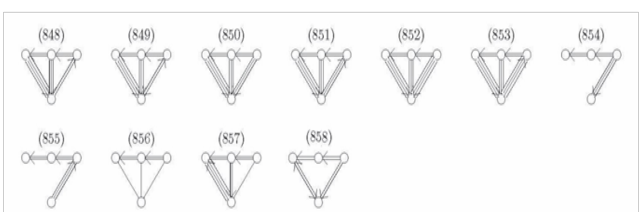

3. Conclusions

In this paper, a complete classification of the Dynkin diagrams is obtained for the particular class of indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_4^{(2)}$. Further study can be carried out to understand properties of imaginary roots and their root multiplicities of other families.

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