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# Fuzzy Dynamic Equations on Time Scales under Generalized Delta Derivative via Contractive-like Mapping Principles

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## Abstract

**Background/Objectives:** The first order nonlinear Fuzzy Dynamic Equations (FDEs) play an important role in recent years to model the dynamic systems with uncertainties and vagueness. The present work deals with obtaining the existence and uniqueness criteria for nonlinear FDEs on time scales which provides a foundation for the nonlinear studies in the field of FDEs. **Methods/Statistical Analysis:** In place of Banach contraction principle, contractive-like mapping principles on partial ordered sets is used as a tool to study the FDEs on time scales under generalized delta derivative. **Findings:** The nonlinear FDEs on time scales using generalized delta derivative are not studied so far. The generalized delta derivative is based on four forms which allow us to obtain new solutions for FDEs with decreasing length of their support. Moreover, the differentiability in third and fourth forms is linked with the concept of switching points. These results include both continuous and discrete FDEs under one framework. **Application/Improvements:** These results are useful to study qualitative and quantitative properties for nonlinear FDEs on time scales which arise in biological, economical and control engineering problems.

**Keywords:** Contractive-Like Mapping, Fuzzy Dynamic Equations, Generalized Delta Derivative, Time Scales

## 1. Introduction

Fuzzy differential equations (Fdes) are appropriate in the modeling of many real-world phenomena. Using the concept of H-derivative defined in[1], Kaleva[2] developed the theory of Fdes. For detailed study on Fdes and their applications we refer to[3-12]. Dynamic equations on time scales[13,14] is an emerging area which unifies effectively both differential and difference equations. An example of this type can be seen in seasonally breeding populations giving rise to new non-overlapping generations. In[15], the author introduced $\Delta g-$derivative and $\Delta g-$integral, studied the fundamental properties of fuzzy set-valued mappings on time scales. In[16], the author introduced $\Delta_{SH}-$derivative and studied the FDEs on time scales.

Preliminary properties and definitions related to fuzzy set-valued mappings, fixed point theorems and calculus on time scale are presented in Section 2. In Section 3, the properties of $\Delta g-$derivative are studied which are necessary for later discussion. In Section 4, with the help of fixed point theorems in[17,18], we establish existence and uniqueness criteria for $\Delta_{1,g}$-solution and $\Delta_{2,g}$-solution of FDEs on time scales.

## 2. Preliminaries

Denote $E^n = \{u : \boldsymbol{R}^n \to [0,\ 1]\}$, u is fuzzy convex, upper semi continuous, normal and support $[u]^0$ is compact. For $v, w \in E^n$

$$D(v,\ w) = \sup_{0 \le q \le 1} d_H([v]^q, [w]^q),$$

Where $d_H$ is the metric defined in 2. For $0 < q \le 1,$ the

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level q− set $[u]^q = \{y \in \boldsymbol{R}^n / u(y) \geq q\} \in P_k(\boldsymbol{R}^n)$ where $P_k(\boldsymbol{R}^n)$ is defined as in 2. For any $v, w \in E^n$ and $\gamma \in R$, $[v+w]^q = [v]^q + [w]^q$, $[\gamma \bullet v]^q = \gamma[v]^q$, $\forall q \in [0,1]$

The partial ordering induced by the set inclusion in $E^n$. i.e.

$v \prec w \Leftrightarrow [v]^q \subseteq [w]^q$, $\forall q \in [0,1]$ v, w $\in E^n$.

The converse of the partial order $\prec$ is denoted by $\succ$.

For any $u, v \in E^n$, $\exists$ a $w \in E^n \ni u = v + w$, $w$ is the H-difference of u and v which is denoted by $u \Theta v$.

**Definition 2.1.**[2] A fuzzy-valued function $G: T \to E^n$ is called Hukuhara differentiable at a point $t_1 \in T$ if $\exists$ a $G'(t_1) \in E^n \ni$ the limits exist in $E^n$

$$G'(t_1) = \lim_{k \to 0^+} \frac{G(t_1 + k)\, \Theta\, G(t_1)}{k} = \lim_{k \to 0^+} \frac{G(t_1)\, \Theta\, G(t_1 - k)}{k}.$$

The element $G'(t_1)$ is the Hukuhara derivative of G at $t_1$ in the metric space ($E^n$, D).

**Definition 2.2.** [18] An altering distance function $\chi: [0, \infty) \to [0, \infty)$ is defined as

(i) $\chi$ is continuous and $\chi(t_2) \geq \chi(t_1)$ for $t_2 \geq t_1$.

(ii) $\chi(t_1) = 0$ □ $t_1 = 0$.

**Definition 2.3.**[18] Let Y be a space with metric $d_1$ and $g_1 : Y \to Y$ be a function. Then $g_1$ is weakly contractive if for any altering distance functions $\chi$ and $\varphi$

$$\chi(d_1(g_1(x_1), g_1(x_2))) \leq \chi(d_1(x_1, x_2)) - \varphi(d_1(x_1, x_2)),\ \forall\ x_1, x_2 \in Y.$$

**Lemma 2.1.** [18] If Y is a partial ordered set with partial ordering $\leq$ and Y is complete metric space with metric $d_1$. Let $g_1 : Y \to Y$ be a mapping, $g_1(x_2) \geq g_2(x_1)$ $\forall x_2 \geq x_1$ and satisfying

$$\chi(d_1(g_1(x_1), g_1(x_2))) \leq \chi(d(x_1, x_2)) - \varphi(d(x_1, x_2)), \quad \text{for } x_1 \geq x_2. \qquad (2.1)$$

Suppose X satisfies either

if a non-decreasing sequence $\{y_m\}_{m \in \boldsymbol{N}}$ is convergent to $y \in Y$, then $\{y_m\} \leq y$, $\forall\ m \in \boldsymbol{N}$, or that $g_1$ is continuous. If $\exists\ y_1 \in Y \ni y_1 \leq g_1(y_1)$, then $g_1$ has a fixed point.

if a non-increasing sequence $\{y_m\}_{m \in \boldsymbol{N}}$ is convergent to $y \in Y$, then $y \leq \{y_m\}$, $\forall\ m \in \boldsymbol{N}$, or that $g_1$ is continuous. If there exists $y_1 \in Y$ such that $y_1 \geq g_1(y_1)$, then $g_1$ has a fixed point.

**Lemma 2.2.** [18] Under the assumption of Lemma 2.1, if every pair of elements of X has an upper bound or a lower bound, then $g_1$ has a unique fixed point. Moreover, if $\bar{x}$ is the fixed point of $g_1$, then $\forall\ x \in X$ $\lim_{k \to \infty} g_1^{\,k}(x) = \bar{x}$.

For basic properties, fundamental results and notations on time scale we follow[14].

## 3. Differentiability and integrability:

Now, we study the results on $\Delta_g$− derivative for FSVF on time scales which are defined in [15]. The $\Delta_g$− derivative given in Definition 13 in [15] can be equivalently written as

**Definition 3.1.** A FSVF $G: \boldsymbol{T} \to E^n$ is called $\Delta_g$−differentiable at $t_1 \in \boldsymbol{T}^k$ if $G_g^{\Delta}(t_1) \in E^n$ if for $0 < k < \delta$, such that

1)

$$\lim_{k \to 0^+} \left( \frac{1}{k - \mu(t_1)} \right) \bullet (G(t_1 + k)\, \Theta\, G(\sigma(t_1))) = \lim_{k \to 0^+} \left( \frac{1}{k + \mu(t_1)} \right) \bullet (G(\sigma(t_1))\ \Theta\, G(t_1 - k)$$
$$= G_g^{\Delta}(t_1)$$

provided the H-differences $G(t_1 + k)\, \Theta\, G(\sigma(t_1))$, $G(\sigma(t_1))\Theta\, G(t_1 - k)$ exists (or)

2)

$$\lim_{k \to 0^+} \left( \frac{-1}{k - \mu(t_1)} \right) \bullet (G(\sigma(t_1))\, \Theta\, G(t_1 + k) = \lim_{k \to 0^+} \left( \frac{-1}{k + \mu(t_1)} \right) \bullet (G(t_1 - k)\, \Theta\, G(\sigma(t_1))$$
$$= G_g^{\Delta}(t_1)$$

provided the H-differences $G(\sigma(t_1)\Theta G(t_1 + k))$, $G(t_1 - k)\Theta G(\sigma(t_1))$ exists (or)

3)

$$\lim_{k \to 0^+} \left( \frac{1}{k - \mu(t_1)} \right) \bullet (G(t_1 + k)\, \Theta\, G(\sigma(t_1))) = \lim_{k \to 0^+} \left( \frac{-1}{k + \mu(t_1)} \right) \bullet (G(t_1 - k)\, \Theta\, G(\sigma(t_1))$$
$$= G_g^{\Delta}(t_1)$$

provided the H-differences $G(t_1 + k)\, \Theta\, G(\sigma(t_1))$, $G(t_1 - k)\, \Theta\, G(\sigma(t_1)))$ exists (or)

$$\lim_{k \to 0^+} \left( \frac{-1}{k - \mu(t_1)} \right) \bullet (G(\sigma(t_1))\ \Theta\, G(t_1 + k) = \lim_{k \to 0^+} \left( \frac{1}{k + \mu(t_1)} \right) \bullet (G(\sigma(t_1))\ \Theta F(t_1 - k)$$
$$= G_g^{\Delta}(t_1),$$

provided the H-differences $G(\sigma(t_1)\ \Theta\, G(t_1+k),\ G(\sigma(t_1)\ \Theta\, G(t_1-k)$ exists.

The element $G^{\Delta_g}(t_1)$ is called the $\Delta g-$derivative of G at $t_1 \in \boldsymbol{T}^{\boldsymbol{k}}$. We say that G is $\Delta_{1,g}-$differentiable, if G is differentiable in form (i) and $\Delta_{2,g}$, $\Delta_{3,g}$, $\Delta_{4,g}$ - differentiable, G is differentiable in (ii),(iii),(iv) forms respectively.

**Remark 3.1.** If T = R, the $\Delta_{1,g}-$ differentiability coincides with the H- derivative defined in [1]. Moreover, the $\Delta_g-$differentiability coincides with the derivative defined in[4]. It is more general than the differentiability introduced in [3] which coincides with (i) and (ii), but it doesn't cover (iii) and (iv).

**Example 3.1.** Let $\boldsymbol{T} = \boldsymbol{N_0^2} = \left\{ \boldsymbol{n^2} : n \in \boldsymbol{N_0} \right\}$ Let $\boldsymbol{G : N_0^2} \to E^1$ defined by $G(t_1) = \sqrt{t_1} \bullet u$, $u \in E^1$ is a triangular fuzzy number then G is $\Delta_{1,g}-$ differentiable at all $t_1 \in \boldsymbol{N_0^2}$.

$$G_g{}^{\Delta}(t_1) = \frac{1}{\mu(t_1)} \bullet (G(\sigma(t_1)\,\Theta\, G(t_1))) = \frac{1}{1+2\sqrt{t_1}} \bullet \left( G\left(\sqrt{t_1}+1\right)^2 \Theta\, G(t_1) \right) \bullet u = \frac{1}{1+2\sqrt{t_1}} \bullet u.$$

**Example 3.2.** Let $G : \boldsymbol{T} \to E^1$ given by $G(t_1) = e_{-p}(t_1, 0) \bullet u$, where $u \in E^1$ is a triangular fuzzy number then G is $\Delta_{2,g}-$differentiable at all $t_1 \in \boldsymbol{T}^{\boldsymbol{k}}$.

$$G_g{}^{\Delta}(t_1) = \frac{1}{\mu(t_1)} \bullet \left( e_{-p}(\sigma(t_1), 0)\ \Theta e_{-p}(t_1, 0) \right) \bullet u = -p(t_1) e_{-p}(\sigma(t_1), 0) \bullet u.$$

Thus, if G is $\Delta_{1,g}-$ differentiable or $\Delta_{2,g}-$ differentiable at $t_1 \in \boldsymbol{T}^{\boldsymbol{k}}$, then G is $\Delta_g-$ differentiable as in Definition 3.1. However, when G is $\Delta_g-$differentiable at $t_1 \in \boldsymbol{T}^{\boldsymbol{k}}$ as in Definition 3.1, then G may or may not be $\Delta_{1,g}$ or $\Delta_{2,g}$ differentiable which can be verified from the following example.

**Example 3.3.** Let $G : \boldsymbol{T} \to E^1$ given by $G(t) = c \bullet |t|$, where $c = (1, 2, 3) \in E^1$ is a triangular fuzzy number. Then G is $\Delta_{1,g}$ –differentiable for all $t \in \boldsymbol{T}^k$ when $t > 0$ and $G^{\Delta_g}(t) = c = (1, 2, 3)$ and G is $\Delta_{2,g}$ – differentiable for all $t \in \boldsymbol{T}^k$ when $t < 0$ and $G^{\Delta_g}(t) = -c = (-3, -2, -1)$ For $t = 0$, neither (i) or (ii) of Definition 3.1. holds. At $t = 0$, the H-differences in (iii) of Definition 3.1. exists.

Let $[c, d]_{\boldsymbol{T}}$ denotes $[c, d] \cap \boldsymbol{T}$.

**Definition 3.2.** Let $G : [c,\ d]_{\boldsymbol{T}} \to E^n$ and $t_0 \in [c, d]_{\boldsymbol{T}}$. The point $t_0$ is said to be a switching point for the $\Delta_g-$differentiable function G, if it satisfies any one of the two conditions below, i.e. for any neighborhood of $t_0$ $\exists$ $t_1 < t_0 < t_2 \in [c,\ d]_{\boldsymbol{T}}$ $\ni$

(I) G is $\Delta_{1,g}-$ differentiable on $(t_1, t_0]_{\boldsymbol{T}}$ and G is $\Delta_{2,g}-$differentiable on $[\sigma(t_0), t_2)_{\boldsymbol{T}}$, i.e. G is non-decreasing on $(t_1, t_0]_{\boldsymbol{T}}$ and non-increasing on $[\sigma(t_0), t_2)_{\boldsymbol{T}}$, or

(II) G is $\Delta_{2,g}-$ differentiable on $(t_1, t_0]_{\boldsymbol{T}}$ and G is $\Delta_{1,g}-$differentiable on $[\sigma(t_0), t_2)_{\boldsymbol{T}}$, i.e. G is non-increasing on $(t_1, t_0]_{\boldsymbol{T}}$ and non-decreasing on $[\sigma(t_0), t_2)_{\boldsymbol{T}}$.

**Theorem 3.1.** Let $G : [c,\ d]_{\boldsymbol{T}} \to E^n$ be a FSVF and $s_0 \in [c,\ d]_{\boldsymbol{T}}$.

(i)If $S_0$ is a switching point of type (I) $\Delta_g-$differentiable of G, then G is $\Delta_{4,g}-$differentiable at $S_0$.

(ii)If $S_0$ is a switching point of type (II) $\Delta_g-$differentiable of G, then G is $\Delta_{3,g}-$differentiable at $S_0$.

**Proof** If $S_0$ is right-scattered and switching point for G of type I, then

$$\lim_{k \to 0^+} \left( \frac{1}{k + \mu(s_0)} \right) \bullet \left( G(\sigma(s_0)\ \ \Theta\, G(s_0 - k) \right), \tag{3.10}$$

$$\lim_{k \to 0^+} \left( \frac{-1}{k - \mu(s_0)} \right) \bullet \left( G(\sigma(s_0)\ \ \Theta\, G(s_0 + k) \right). \tag{3.11}$$

The limit in (3.10) exists if G is $\Delta_{1,g}-$ differentiable or $\Delta_{4,g}-$ differentiable and the limit in (3.11) exists if G is $\Delta_{2,g}-$ or $\Delta_{4,g}-$ differentiable. Hence G is $\Delta_{4,g}-$ differentiable. In a similar way, we can prove (ii).

**Definition 3.3.** [15] The $\Delta_g-$ integral of $G : \boldsymbol{T} \to E^n$ on $J \subset \boldsymbol{T}$, defined level wise by

$$\left[ \int_J G(s)\Delta s \right]^q = \int_J G_q(s)\Delta s = \left\{ \int_J g(s)\Delta s : g \in S_{F_q}(J) \right\},$$

where $S_{F_q}(J)$ the set of all $\Delta_g-$ integrable sectors of $G_q$ on J.

For properties on $\Delta_g-$ integral, we refer to [15].

## 4. FDEs on time scales

Now, we focus our attention on the following nonlinear FDE on time scale

$$z^{\Delta}(t)=G(t,z(t)),\ \ z(t_0)=z_0,\ t\in[c,\ d]_{\mathbf{T}}, \tag{4.1}$$

$\Delta$ denotes the $\Delta_g-$ derivative. Throughout $G:[c,d]_{\mathbf{T}}\times E^n\to E^n$ is a nonlinear function which is rd-continuous and $t_0\in[c,\ d]_{\mathbf{T}}$, $u_0\in E^n$. Eq. (4.1) is called FIVP on time scales.

**Definition 4.1.** Let $C_{rd}([c,\sigma(d))]_T, E^n$ be the rd-continuous fuzzy functions.
(i)A solution $z(t) \in C_{rd}([c,\sigma(d))]_T, E^n)$ is called a $\Delta_g-$differentiable solution for (4.1) if $z(t)$ is an anti-derivative of $G(t,z(t)$ satisfying (4.1).

A solution for (4.1) is called
(ii) (1)-solution if it is $\Delta_{1,g}-$ differentiable.
(iii) (2)-solution if it is $\Delta_{2,g}-$differentiable.
Consider the partial order on $C=C_{rd}([c,\sigma(d))]_T, E^n)$ as $g_1\prec g_2 \Leftrightarrow g_1(t_1)\prec g_2(t_1)$ $g_1,g_2\in C$ and $\forall$ $t_1\in[c,\ d]_{\mathbf{T}}$.
Clearly, $C_{rd}([c,\sigma(d))]_T, E^n)$ $\prec$ is partial ordered set.

**Lemma 4.1.** [18] The following results hold on $C_{rd}([c,\sigma(d))]_T, E^n)$:
(i)If $\{f_m\}_{m\in\mathbf{N}}\subset C$ is nondecreasing, with partial ordering $\prec$ such that $f_m\to f$ in C, then $f_m\prec f$ $\forall$ $m\in\mathbf{N}$.

(ii)If $\{f_m\}_{m\in\mathbf{N}}\subset C$ is nonincreasing, with partial ordering $\prec$ such that $f_m\to f$ in C, then $\forall$ $m\in\mathbf{N}$.

**Lemma 4.2.** Let G be rd-continuous.
(i)A fuzzy function $z\in C$ is called a $\Delta_{1,g}-$ differentiable solution to (4.1) iff it satisfies the integral equation

$$z(t_1)=z_0+\int_{t_0}^{t_1}G(s,z(s)\ \Delta s,\quad t_1\in[c,\ \sigma(d))]_{\mathbf{T}}. \tag{4.2}$$

(ii)A fuzzy function $z\in C$ is called a $\Delta_{2,g}-$ differentiable solution to (4.1) iff it satisfies the integral equation

$$z_0=z(t_1)+(-1)\int_{t_0}^{t_1}G(s,z(s)\ \Delta s,\ \forall\ t_1\in[c,\sigma(d))]_{\mathbf{T}}, \tag{4.3}$$

or

$$z(t_1)=z_0\ \Theta\left((-1)\int_{t_0}^{t_1}G(s,z(s)\ \Delta s\right),\ \forall\ t_1\in[c,\sigma(d))]_{\mathbf{T}}, \tag{4.4}$$

**Definition 4.2.** A solution for (4.1) is said to be a lower solution if the fuzzy function $\eta\in C$ satisfies

$$\eta^{\Delta}(t_1)\prec G(t_1,\ \eta(t_1)),\ t_1\in[c,\ \sigma(d)]_{\mathbf{T}},\ \ \eta(0)\prec z_0.$$

If $\eta$ is $\Delta_{1,g}-$ differentiable, then $\eta$ is said to be a lower $\Delta_{1,g}-$ differentiable and if $\eta$ is $\Delta_{2,g}-$differentiable, then it is lower $\Delta_{2,g}-$ differentiable.

A solution for (4.1) is said to be a upper solution if the fuzzy function $\eta\in C$ satisfies

$$\eta^{\Delta}(t_1)\succ F(t_1,\eta(t_1)),\ t_1\in[c,\sigma(d)]_{\mathbf{T}},\ \ \eta(0)\succ z_0.$$

If $\eta$ is $\Delta_{1,g}-$ differentiable (respectively $\Delta_{2,g}-$ differentiable), then $\eta$ is said to be a upper $\Delta_{1,g}-$ differentiable solution (respectively, a upper $\Delta_{2,g}-$ differentiable solution).

**Theorem 4.1.** (Local Existence and Uniqueness theorem) Let $G:[c,\ d]_{\mathbf{T}}\times E^n\to E^n$ be rd- continuous. If there exists a lower $\Delta_{1,g}-$ differentiable solution $\eta \in C_{rd}([c,\sigma(d))]_T, E^n)$ for (4.1) and

(i)G is non-decreasing w.r.to second variable, i.e. for $v_1\succ w_1$ then $G(t_1,v_1)\succ G(t_1,w_1)$

(ii)For comparable elements $G$ is weakly contractive, i.e. for altering distance functions $\chi$ and $\varphi$,

$$\chi(D(G(t_1,v_1),G(t_1,w_1)))\le\ \chi(D(v_1,w_1))-\varphi(D(v_1,w_1)),\ \text{f}\ v_1\succ w_1. \tag{4.5}$$

Then a unique $\Delta_{1,g}-$ differentiable solution z exists for (4.1) on $[c,\ \sigma(d)]_{\mathbf{T}}$.

**Proof** Define the operator $A_1:C\to C$ by

$$[A_1z](t_1)=z_0+\int_{0}^{t_1}G(s,z(s)\ \Delta s,\quad t_1\in[c,\sigma(d))]_{\mathbf{T}}.$$

From Lemma 4.1 (i), $z\in C$ is the solution of (4.1), if $z\in C$ is the fixed point of $A_1$.

Define $D_\varepsilon$, a metric on C by

$$D_\varepsilon(v_1,\ w_1)=\sup_{s\in[c,\sigma(d)]_{\mathbf{T}}}\{D(v_1(s),w_1(s))\,e_{-\varepsilon}(s,0)\},\quad v_1,w_1\in C,$$

Where $\varepsilon>0$ large enough such that $\dfrac{1-e_{-\varepsilon}(T,0)}{\varepsilon}<1$.

This metric is equivalent to metric D, because $D_\varepsilon(v_1,\ w_1)\le D(v_1,\ w_1)\le e_\varepsilon(T,\ 0)D_\varepsilon(v_1,w_1)$ , $\forall$ $v_1,w_1\in C$
However, $C_{rd}([c,\sigma(d))]_T, E^n)$, $D_\varepsilon$ is a complete metric space.

From assumption (i) and from Lemma 2.1 we have

$$[A_1v_1](t_1)=z_0+\int_0^{t_1}G(s,v_1(s)\ \Delta s\succ z_0+\int_0^{t_1}G(s,w_1(s)\ \Delta s=[A_1w_1](t_1)$$

whenever $v_1\succ w_1$ and $t_1\in[c,\sigma(d))]_{\mathbf{T}}$. Hence the operator $A_1$ is non-decreasing. Now from (ii),

$$\chi(D(G(t_1, v_1), G(t_1, w_1))) \leq \chi(D(v_1, w_1)), \ \forall \ v_1 \succ w_1. \qquad (4.6)$$

In a contrary, Assume that

$D(v_1, w_1) < D(G(t_1, v_1), G(t_1, w_1)), \quad \forall \ v_1 \succ w_1.$

Since $\chi$ is non-decreasing we have $\chi(D(v_1, w_1)) \leq \chi(D(G(t_1, u_1), G(t_1, v_1))).$

From (4.6), $\chi(D(v_1, w_1)) = \chi(D(G(t_1, v_1), G(t_1, w_1))),$ $\forall v_1 \succ w_1.$

From (ii), $\varphi(D(v_1, w_1)) = 0$ which implies $D(v_1, w_1) = 0$ and hence $\chi(D(G(t_1, v_1), G(t_1, w_1))) = 0.$ From Definition 2.2., we get $D(G(t_1, v_1), G(t_1, w_1)) = 0$ which is a contradiction. For $v_1 \succ w_1$, consider

$$D_\varepsilon(A_1 v_1, A_1 w_1)(t_1) = \sup_{t_1 \in [c,\ \sigma(d)]_{\boldsymbol{T}}} \{ D( [A_1 v_1](t_1), [A_1 w_1](t_1)) e_{-\varepsilon}(t_1, 0) \}$$

$$= \sup_{t_1 \in [c, \sigma(d)]_{\boldsymbol{T}}} \left\{ D\left( \int_0^{t_1} G(s, v_1(s))\ \Delta s,\ \int_0^{t_1} G(s, w_1(s))\ \Delta s \right) e_{-\varepsilon}(t_1, 0) \right\}$$

$$\leq \sup_{t_1 \in [c, \sigma(d)]_{\boldsymbol{T}}} \left\{ \int_0^{t_1} D(G(s, v_1(s)),\ G(s, w_1(s)))\Delta s\ e_{-\varepsilon}(t_1, 0) \right\}$$

$$\leq \sup_{t_1 \in [c,\ \sigma(d)]_{\boldsymbol{T}}} \left\{ D_\varepsilon(v_1, w_1) \int_0^{t_1} e_\varepsilon(s, 0)\ \Delta s\ e_{-\varepsilon}(t_1, 0) \right\}$$

$$\leq D_\varepsilon(v_1, w_1) \sup_{t_1 \in [c, \sigma(d)]_{\boldsymbol{T}}} \left\{ \left( \frac{e_\varepsilon(t_1, 0) - 1}{\varepsilon} \right) e_{-\varepsilon}(t_1, 0) \right\}$$

$$= D_\varepsilon(v_1, w_1) \sup_{t_1 \in [c,\ \sigma(d)]_{\boldsymbol{T}}} \left\{ \left( \frac{1 - e_{-\varepsilon}(t_1, 0)}{\varepsilon} \right) \right\} = \left( \frac{1 - e_{-\varepsilon}(T_1, 0)}{\varepsilon} \right) D_\varepsilon(v_1, w_1)$$

Therefore

$$D_\varepsilon(A_1 v_1, A_1 w_1) \leq \left( \frac{1 - e_{-\varepsilon}(T_1, 0)}{\varepsilon} \right) D_\varepsilon(v_1, w_1) \quad \forall v_1,\ w_1 \in C_{rd}([c, \sigma(d)])_{\boldsymbol{T}}$$

Hence for altering distance function $\beta$,

$$\beta(D_\varepsilon(A_1 v_1, A_1 w_1)) \leq \beta\left( \left( \frac{1 - e_{-\varepsilon}(T_1, 0)}{\varepsilon} \right) D_\varepsilon(v_1, w_1) \right)$$
$$= \beta(D_\varepsilon(v_1, w_1)) - \left[ \beta(D_\varepsilon(v_1,\ w_1)) - \beta\left( \left( \frac{1 - e_{-\varepsilon}(T_1, 0)}{\varepsilon} \right) D_\varepsilon(v_1, w_1) \right) \right],$$

holds. Then if $\varphi(t_1) = \beta(t_1) - \beta\left( \left( \frac{1 - e_{-\varepsilon}(T_1, 0)}{\varepsilon} \right) t_1 \right)$ for $v_1 \succ w_1$ it follows that,

$$\beta(D_\varepsilon(A_1 v_1, A_1 w_1)) \leq \beta(D_\varepsilon(v_1,\ w_1)) - \phi(D_\varepsilon(v_1, w_1)).$$

By the existence of lower $\Delta_{1,g}$– differentiable solution and Lemma 4.2 (i),

$$\eta(t_1) = \eta(0) + \int_0^{t_1} \eta^\Delta(s)\Delta s \prec z_0 + \int_0^{t_1} G(s, \eta(s))\ \Delta s = [A_1 \eta](t_1) \qquad t_1 \in [c, \sigma(d))]_{\boldsymbol{T}}.$$

Thus $\eta \prec A_1 \eta$. Hence $A_1$ satisfies all hypotheses of Lemma 2.1 and Lemma 2.2, and therefore $A_1$ has the unique fixed point which itself is the unique $\Delta_{1,g}$– solution for (4.1).

**Example 4.1.** Consider the FIVP $z^\Delta(t) = G(t, z(t)),$ where

$$G(t_1, z(t_1)) = \begin{bmatrix} \tan^{-1}(u_1(t_1)) \\ u_2(t_1) \end{bmatrix},\ t_1 \in [0, 5]_{\boldsymbol{T}},\ \ z(0) = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \in E^2,$$

We claim this FIVP has unique solution for T = R Clearly

$$D(G(t, v_1), G(t, w_1)) \leq \sup_{r \in \boldsymbol{R}} \frac{1}{1 + r^2} D(v_1, w_1)$$

Consider the altering distance function $\chi(t_1) = t_1$ and $\varphi(t_1) = \frac{r^2}{1 + r^2} t_1$, for some $r \in \boldsymbol{R}$. Then all the assumptions in Theorem 4.1 are fulfilled and therefore the FIVP has unique solution.

**Theorem 4.2.** Let $G : [c,\ d]_{\boldsymbol{T}} \times E^n \to E^n$ be rd- continuous.

If there exists a lower $\Delta_{2,g}$– differentiable solution $\eta \in C_{rd}([c, \sigma(d))]_T, E^n)$, for (4.1). Let G be such that:

(i)

$$diam([u_0]^q) \geq diam\left( \left[ \int_0^{t_1} G(s, z(s))\ \Delta s \right]^q \right), \quad \forall\ q \in [0, 1]$$

(ii) G is non-decreasing in the second variable, i.e. if $v_1 \succ w_1$ then $G(t_1, v_1) \succ G(t_1, w_1)$

(iii) For comparable elements, G is weakly contractive i.e. for altering distance functions $\chi$ and $\varphi$, satisfying (4.5).

Then a unique $\Delta_{2,g}$– differentiable solution z exists for (4.1) on $[c, \sigma(d)]_{\boldsymbol{T}}$.

**Proof** Define the operator $A_2 : C \to C$ by

$$[A_2 z](t_1) = z_0 \Theta \left( (-1) \int_{t_0}^{t_1} G(s, z(s)) \Delta s \right), \; t_1 \in [c, \sigma(d))]_{\boldsymbol{T}}.$$

From Lemma 4.2 (ii), z is a solution of (4.1). Also, the operator $A_2$ is non-decreasing for $u \succ v$. In a similar way to Theorem 4.1., $A_2$ fulfills all assumptions of Lemma 2.1. and hence from Lemma 2.2. $A_2$ has the unique $\Delta_{2,\,g}$– solution for (4.1).

**Theorem 4.3.** Theorems 4.1, 4.2 are also valid if we replace the existence of lower $\Delta_{1,\,g}$– differentiable solution ($\Delta_{2,\,g}$– differentiable solution) to (4.1) by an upper

$\Delta_{1,\,g}$–solution ( $\Delta_{2,\,g}$– solution).

**Proof** If η is an upper $\Delta_{2,\,g}$– differentiable solution for (4.1), then

$$\eta(t_1) = \eta(0) + \int_0^{t_1} \eta^{\Delta}(s) \Delta s \succ z_0 + \int_0^{t_1} G(s, \eta(s)) \Delta s = [A_1 \eta](t_1) \qquad t_1 \in [c, \sigma(d))]_{\boldsymbol{T}}.$$

Similarly, If η is an upper $\Delta_{2,\,g}$– differentiable solution for (4.1), we have

$$\eta(t_1) = \eta(0) \; \Theta \left( (-1) \int_0^{t_1} \eta^{\Delta}(s) \Delta s \right) \succ y_0 \; \Theta \left( (-1) \int_0^{t_1} F(s, \eta(s)) \Delta s \right) = [A_2 \eta(t_1)], \quad t_1 \in [c, \sigma(d))]_{\boldsymbol{T}}.$$

Thus $\eta \succ A_1 \eta$ and $\eta \succ A_2 \eta$. Hence $A_1$ and $A_2$ satisfies all hypotheses of Lemma 2.1 and Lemma 2.2 and therefore $A_1$, $A_2$ has a unique solution in C.

## 5. References


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