

Slope Number On Complete Graphs

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Abstract

Objective: A straight line drawing is a mapping of an edge into a straight line segment. The minimum number of distinct slopes used in a straight line drawing of a graph G is called the slope number of the graph G . In this paper the slope number of complete graph is studied elaborately. **Methods:** This optimization problem is NP-Hard for any arbitrary graph. A canonical way of drawing of a complete graph is an existing one. In present paper, we consider the edges of a complete graph are straight line segments in order to obtain the number of slopes. **Findings:** This paper interprets the characterization of slopes in complete graph according to an odd and even number of edges and investigated in detail. Moreover, the slope number of a complete graph is compared with the chromatic number of complete graph and the results are observed. **Applications/Improvement:** Slope number is one of the quality measures of graph drawing. It is used to find out different layout methods for the same graph.

Keywords: Chromatic Number, Complete Bipartite Graph, Complete Graph, Slope Number, Straight Line Drawing

1. Introduction

Graph theory is the fascinated and rapid growing area in various fields. It has many practical applications in various disciplines namely, biology, computer science, economics, engineering, mathematics, medicine and social network. A graph can be represented by a diagram. A graph describes any physical situation which implicates discrete objects and relates them. Graph drawing is a familiar concept of graph theory and it has many quality measures and one among them is a slope number. The slope number problem was first introduced by Wade and Chu in 1994. They proved that the slope number of complete graph of k_n is n and executed an algorithm¹. Moreover, Jamison discussed the characterization of slopes for regular polygons².

Some related work is outlined from the literature. Ambruset al., were the first to investigate the slope parameter for several graphs such as tree, outerplanar graph, etc³. Any cubic graph can be drawn in the plane using four basic slopes $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{-\pi}{4}\right\}$ ⁴. The slope number is at least $\left\lceil \frac{d}{2} \right\rceil$ only when the graph has a vertex of degree d . Dujmovic et al., suggested if the slope number of a graph with bounded $\Delta(G)$ could be arbitrarily large⁵. Pach

and Pavolyogi and Barat et al., explained individually that the counting argument that graphs with $\Delta(G) = 5$ exceeds the number of graphs with a fixed number of slopes^{5,6}. It was exposed that five slopes can be used in drawing of a cubic (3 regular graph)⁷. Layouts with few slopes and few bends have been elaborately investigated in "Graph Drawing"⁸. Ferber and Jurgensen have considered drawings of lattices and posets with few slopes⁹. Since the slope number is one of the quality measures for graph drawing and gives the nice structural outlook of graphs, we would like to concentrate on the slope number problem.

2. Preliminaries

Consider a simple, connected and undirected graph. A graph is a collection of vertices $V(G)$ and edges $E(G)$ between them. A sub-graph H of G is a graph consists of $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is said to be complete in which every pair of vertices is connected by an edge. In a straight line drawing, the vertices and edges are represented as points and straight line segments respectively. In this paper the slope number of complete graphs is studied in a descriptive manner. The chromatic number of k_n is also compared with the slope number of k_n .

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3. Overview of the paper

To get a pictorial diagram of a graph G , points and lines are used as vertices and edges respectively. We have the following characteristics of drawing of complete graphs with a straight line between each pair of vertices. In view of this line, the characterization of slope number of k_n is investigated and studied elaborately as per the slopes of an edge.

3.1 Slope Number

The minimum number of distinct edge slopes required to draw the graph G is called the slope number. It is written as $sl[G]$.

From the Figure 1, the slope number of G is 4 i.e., $sl[G] = 4$.

i.e., Let s_1 be the slope of an edge v_1v_6 . The edges v_2v_5 and v_3v_4 can be drawn parallel to the slope s_1 . In the same way, we can define the other slopes.

The following lemmas and theorems contribute the fundamental concept of the main results.

3.2 Lemma¹

In a drawing of a complete graph k_n all edges that share the same vertex must have different slopes.

3.3 Lemma¹

$$sl[k_n] \geq n, \text{ for } n \geq 3.$$

3.4 Lemma¹

$$sl[k_n] \leq n.$$

3.5 Theorem¹

$$sl[k_n] = \begin{cases} 0, & \text{if } n = 1 \\ 1, & \text{if } n = 2 \\ n & \text{if } n \geq 3 \end{cases}$$

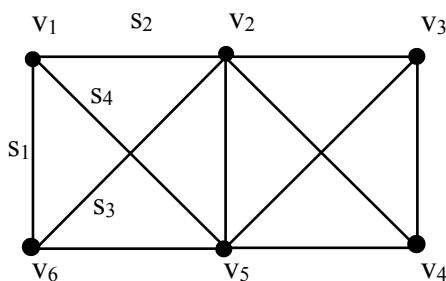


Figure 1. slope number is 4.

4. Main Results

The characterizations on complete graphs are as follows.

4.1 Theorem

For every complete graph G , the slope of the graph k_{4n-1} is odd if and only if it has odd number of edges.

Proof:

Assume that the slope of the graph of k_{4n-1} is odd. To prove that the edges of k_{4n-1} is odd. Let the slopes of the graph be $S = \{s_1, s_2, \dots, s_n\} \in k_{4n-1}$. The maximum degree of k_{4n-1} is $4n - 2$. Clearly, the vertices of a graph G have even degree. Since it is a complete graph, it has $\frac{n(n-1)}{2}$ edges.

Here the edges of k_{4n-1} are $\frac{(4n-1)(4n-2)}{2} = 8n^2 - 6n + 1$, which is odd.

Conversely, if given G is on $4n - 1$ vertices with odd number of edges. Atmost two edges with a common vertex V have different slope. Therefore, every vertex of even degree is an end point of even line segments. The vertices $V(G) = \{v_1, v_2, \dots, v_n\}$ are labelled in an order. Also, $E(G) = \{e_1, e_2, e_3, \dots\}$ be the sequence of edges of k_{4n-1} , where e_i is associated with $v_i v_{i+1}$. By lemma [3.2], In a drawing of a k_n , all the edges share a same vertex must have different slopes. Choose the vertex v_i . The edges incident to the vertex v_i have $4n - 2$ slopes. Next, choose the vertex v_{i+1} . Adding an edge to the vertex v_{i+1} gives one more slope. Hence no more vertices can be chosen other than the vertices v_i and v_{i+1} , since the edges incident to other vertices will be parallel to one of the slopes of v_i and v_{i+1} .

$\therefore sl[k_{4n-1}] = (4n - 2) + 1$ An edge incident to v_{i+1} and not parallel to v_i .

$$= (4n - 2) + 1$$

$$= 4n - 1 \text{ which is odd.}$$

Hence, the proof.

4.2 Theorem

For every complete graph G , the slope of the graph k_{4n} is even if and only if it has even number of edges.

Proof:

Assume that the slope of the graph of k_{4n} is even. To prove that the edges of k_{4n} is even. Let the slopes of the graph be $S = \{s_1, s_2, \dots, s_n\} \in k_{4n}$. The maximum degree of k_{4n} is $(4n - 1)$. Clearly, the vertices of a graph G have odd degree. Since it is a complete graph, it has $\frac{n(n-1)}{2}$ edges.

Here the edges of k_{4n} are $\frac{(4n)(4n+1)}{2} = 8n^2 - 2n$, which is even.

Conversely, if given G is on $4n$ vertices with even number of edges. Atmost two edges with a common vertex V have different slope. Therefore, every vertex of odd degree is an end point of odd line segments. The vertices $V(G) = \{v_1, v_2, \dots, v_n\}$ are labelled in an order. Also, $E(G) = \{e_1, e_2, e_3, \dots\}$ be the sequence of edges of k_{4n} , where e_i is associated with $v_i v_{i+1}$. By lemma [3.2], In a drawing of a k_{4n} , all the edges share a same vertex must have different slopes. Choose the vertex v_i . The edges incident to the vertex v_i have $4n - 1$ slopes. Next, choose the vertex v_{i+1} . Adding an edge to the vertex v_{i+1} gives one more slope, Hence no more vertices can be chosen other than the vertices v_i and v_{i+1} , since the edges incident to other vertices will be parallel to one of the slopes of v_i and v_{i+1} .

$\therefore \text{sl}[k_{4n}] = (4n - 1) + 1$ An edge incident to v_{i+1} and not parallel to v_i .

$$= (4n - 1) + 1$$

$$= 4n \text{ which is even}$$

Hence, the proof.

4.3 Theorem

For every complete graph G , the slope of the graph k_{4n+1} is odd if and only if it has even number of edges.

Proof:

Assume that the slope of the graph of k_{4n+1} is odd. To prove that the edges of k_{4n+1} is even. Let the slopes of the graph be $S = \{s_1, s_2, \dots, s_n\} \in k_{4n+1}$. The maximum degree of k_{4n+1} is $4n$. Clearly, the vertices of a graph G have even degree. Since it is a complete graph, it has $\frac{n(n-1)}{2}$ edges.

Here the edges of k_{4n+1} are $\frac{(4n)(4n+1)}{2} = 8n^2 + 2n$, which is even.

Conversely, if given G is on $4n + 1$ vertices with even number of edges. Atmost two edges with a common vertex V have different slope. Therefore, every vertex of even degree is an end point of even line segments. The vertices $V(G) = \{v_1, v_2, \dots, v_n\}$ are labelled in an order. Also, $E(G) = \{e_1, e_2, e_3, \dots\}$ be the sequence of edges of k_{4n+1} , where, e_i is associated with $v_i v_{i+1}$. By lemma [3.2], in a drawing of a k_{4n} , all the edges share a same vertex must have different slopes. Choose the vertex v_i . The edges incident to the vertex v_i have $4n$ slopes. Next, choose the vertex v_{i+1} . Adding an edge to the vertex v_{i+1} gives one more slope. Hence, no more vertices can be chosen other than the vertices v_i

and v_{i+1} , since the edges incident to other vertices will be parallel to one of the slopes of v_i and v_{i+1} .

$\therefore \text{sl}[k_{4n+1}] = 4n + 1$ An edge incident to v_{i+1} and not parallel to v_i .

$$= 4n + 1$$

$$= 4n + 1 \text{ which is odd}$$

Hence, the proof.

4.4 Theorem

For every complete graph G , the slope of the graph k_{4n+2} is even if and only if it has odd number of edges.

Proof:

Assume that the slope of the graph of k_{4n+2} is even. To prove the edges of k_{4n+2} is odd. Let the slopes of the graph be $S = \{s_1, s_2, \dots, s_n\} \in k_{4n+2}$. The maximum degree of k_{4n+2} is $4n + 1$. Clearly, the vertices of a graph G have odd degree.

Since it is a complete graph, it has $\frac{n(n-1)}{2}$ edges. Here the edges of k_{4n+2} are $\frac{(4n+1)(4n+2)}{2} = 8n^2 + 6n + 1$, which is odd.

Conversely, if given G on $4n + 1$ vertices with odd number of edges. Atmost two edges with a common vertex V have different slope. Therefore, every vertex of odd degree is an end point of odd line segments. The vertices $V(G) = \{v_1, v_2, \dots, v_n\}$ are labelled in an order. Also, $E(G) = \{e_1, e_2, e_3, \dots\}$ be the sequence of edges of k_{4n+2} , where e_i is associated with $v_i v_{i+1}$. By lemma [3.2], In a drawing of a k_n , all the edges share a same vertex must have different slopes. Choose the vertex v_i . The edges incident to the vertex v_i have $4n + 1$ slopes. Next, choose the vertex v_{i+1} . Adding an edge to the vertex v_{i+1} gives one more slope. Hence no more vertices can be chosen other than the vertices v_i and v_{i+1} , since the edges incident to other vertices will be parallel to one of the slopes of v_i and v_{i+1} .

$\therefore \text{sl}[k_{4n+2}] = (4n + 1) + 1$ An edge incident to v_{i+1} and not parallel to v_i .

$$= (4n + 1) + 1$$

$$= 4n + 2, \text{ which is even.}$$

Hence, the proof.

5. Chromatic Number of a Graph

A graph colouring assigns a colour to each vertex in a way so that no edge has both vertices the same colour. The chromatic number of a graph is the least number of colours needed for a colouring of G . It is denoted by $\chi[G]$. It is a well-known fact that the chromatic number of k_n is n .

5.1 Observation

For any complete graph K_n on n vertices, $\chi[K_n] = \text{sl}[K_n] = n$.

The following Figure 2a,2b illustrates the chromatic number and slope number of the complete graph on 8 vertices.

Remark

In line of thought the complete graph K_n is similar to the complete bipartite graph $K_{m,n}$. But there is a significant variation to verify the slope number of K_n is even if and only if it has odd number of edges whereas the slope number of $K_{m,n}$ is even if and only if it has even number of edges. So special attention need to check the behaviour of the complete bipartite graph in this case. Further research work on characterization of complete bipartite graph will be carried out and investigated in future. The illustration of complete bipartite graph is shown in the Figure 3a, 3b, 3c.

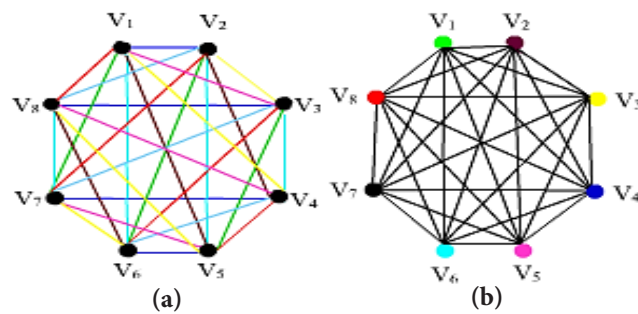


Figure 2 (a) Slope number of $K_8 = 8$. (b) Chromatic number of $K_8 = 8$.

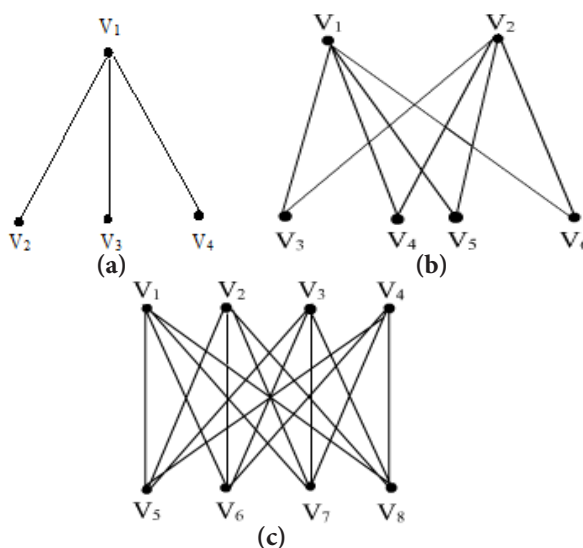


Figure 3 (a) Odd slopes - odd edges. (b) Even slopes - (c) Odd slopes - even edges.

6. Conclusion

Thus we characterized and investigated results on slope number of complete graphs according to odd and even number of edges. Finally, the chromatic and slope number of K_n is observed. We have also discussed a few facts that whether similar results hold for complete bipartite graph. This slope number problem is worth considering for cayley graphs and non-cayley graphs. Furthermore, an extensive way towards the parallel architecture of interconnection networks relevant to this problem is our future scope. Also we are focussing on star chromatic number of the interconnection networks^{10,11}.

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