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Quasi Affine Generalized Kac Moody Algebras $QAGGD_3^{(2)}$: Dynkin diagrams and root multiplicities for a class of $QAGGD_3^{(2)}$

A. Uma Maheswari*

Quaid-e-millath Government College for Women (Autonomous), Chennai – 600 002, Tamil Nadu, India;
umashiva2000@yahoo.com

Abstract

Objectives: To define the class of quasi affine generalized Kac-Moody algebras $QAGGD_3^{(2)}$, completely classify the non isomorphic, connected Dynkin diagrams associated with $QAGGD_3^{(2)}$ and compute some root multiplicities for this family. **Methods:** The representation theory of Kac-Moody algebras is applied to compute the multiplicities of roots for a quasi affine family in $QAGGD_3^{(2)}$. **Findings:** The quasi affine generalized Kac-Moody algebras associated with symmetrizable Generalized Generalized Cartan Matrices (GGCM) of quasi affine type, obtained from the affine family $D_3^{(2)}$, are defined; The connected, non-isomorphic Dynkin diagrams associated with this particular family are completely classified'; Multiplicities of roots of a class GKM algebras $QAGGD_3^{(2)}$, with one simple imaginary root are then determined using the representation theory of Kac Moody algebras; **Application:** Generalized Kac-Moody algebras find interesting applications in bosonic string theory, classifications in vertex operator theory , monstrous moonshine theory etc.

Keywords: Dynkin Diagram, Generalized Kac Moody Algebras (GKM), Imaginary Roots, Quasi Affine, Root Multiplicity

1. Introduction

Generalized Kac Moody algebras (GKM algebras), introduced in[1] differs from Kac Moody algebras mainly in the existence of imaginary simple roots in GKM algebras. The basic results on structure and representation theory of Kac moody algebras, can be extended to GKM algebras also; Root multiplicities for some extended hyperbolic Generalized Kac Moody algebras were computed in[2]. Root properties and root multiplicities of some GKM algebras, extending the GCM of finite, affine and hyperbolic types were studied in[3-8].

In[9-12], closed form root multiplicity formulae for GKM algebras were obtained. In[13], multiplicities of simple imaginary roots for the Borcherds algebra $gII_{9,1}$ were determined. In[14,15], the family EB_2 was studied. In[16], root multiplicities for the quasi affine generalized Kac Moody algebras $QAGGD_3^{(2)}$ were computed; In[12], the dimension formula, applied to various classes of graded Lie algebras were derived in [17-19]. In[20], quasi affine Kac Moody algebras, belonging to the indefinite class of Kac Moody algebras was defined and the quasi affine algebras $QAC_2^{(1)}$ was studied. In[21-24], extended hyperbolic type of indefinite Kac Moody algebras were introduced, wherein the structure of $EHA^{(1)}_1$ and $EHA^{(2)}_2$ were studied.

In this paper quasi affine generalized Kac Moody algebras $QAGGD_3^{(2)}$ are defined; the general form of non isomorphic, connected Dynkin diagrams associated with the symmetrizable GKM algebras $QAGGD_3^{(2)}$ are given; We then consider a specific family $QAGGD_3^{(2)}$ of GGCM of quasi affine type, with one imaginary simple root, which are obtained from the affine GCM $D_3^{(2)}$. Finally we explicitly compute the root multiplicities in these GKM algebras which are obtained as extensions of the affine family$D_3^{(2)}$.

2. Preliminaries

We recall the basic definitions and results of GKM algebras in[1,11,18,25-27].

Definition 2.1[1]: A Borcherds Cartan matrix (BKM) is a real matrix $A = (a_{ij})_{i,j=1}^{n}$ satisfying the conditions : i) $a_{ij} = 0$

* *Author for correspondence*

or $a_{ij} \le 0$ for all i ε I ii) $a_{ij} \le 0$ for i≠j, a_{ij} ε Z if a_{ii} =2 , iii) a_{ij} = 0 implies a_{ji} = 0.`

We assume that the BKM is symmetrizable. Let the real simple and real imaginary roots be denoted by I^{re} = {i ε I / a_{ii} =2 } and I^{im} = { i ε I / $a_{ii} \le 0$}; Let the charge m = { m_i ε $Z_{>0}$ / i ε I } be a collection of positive integers such that m_i=1 for all i ε I^{re}.

The Generalized Kac-Moody algebra g=g(A, m) associated with a symmetrizable BKM matrix A = $(a_{ij})_{i,j=1}^{n}$ of charge m = (m_i / i ε I}is the Lie algebra generated by the elementsh$_i$, d_i, e_{ik}, f_{ik}, i ε I ,k=1,…,m_i with the following defining relations :

$[h_i,h_j] = [d_i,d_j] = [h_i,d_j] = 0$, $[h_i,e_{jl}] = a_{ij}e_{jl}$, $[h_i,f_{jl}] = -a_{ij}f_{jl}$,
$[d_i,e_{jl}] = \delta_{ij}\, e_{jl}$, $[d_i,f_{jl}] = -\delta_{ij}\, f_{jl}$, $[e_{ik},f_{jl}] = \delta_{ij}\delta_{kl}\, h_i$

$$(ad\ e_{ik})^{(1-a_{ij})}(e_{jl}) = (ad\ \ f_{ik})^{(1-a_{ij})}(f_{jl}) = 0 \ \ if\ \ a_{ii} = 2, i \ne j,$$

$$[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0\ \ if\ \ a_{ij} = 0, (i, j \in I, k = 1, ..., m_i, l = 1, ..., m_j).$$

The subalgebra h = (⊕ Ch_i) ⊕ (⊕ Cd_i)is called the Cartan subalgebra of g. For each i ε I, we define a linear functional α_i ε h*, by α_i (h_j)=a_{ij}, α_i (d_j)=$a_{ij}\delta_{ij}$,i,j εI. α_i's are called the simple roots of g.

The GKM algebra g=g(A, m) has the root space decomposition

$$g = \underset{\alpha \in Q}{\oplus} g_\alpha, \text{ where } g_\alpha = \{x \in g / [h, x] = \alpha(h)x, for\ all\ h \in H\}.$$

Let $Q+ = \sum_{i=1}^{n} Z_+\alpha_i$. Q has a partial ordering "≤" on h* defined by $\alpha \le \beta$ if β - α ∈ Q_+, where α, β ∈ Q.

Definition 2.2[27]: In Kac Moody algebras the Dynkin diagrams are defined as follows: To every GCM A is associated a Dynkin diagram S(A) defined as follows : S(A) has n vertices and vertices i and j are connected by max {|a_{ij}|,|a_{ji}|} number of lines if a_{ij}. $a_{ji} \le 4$ and there is an arrow pointing towards I if |a_{ij}|>1. If $a_{ij}.a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair (|a_{ij}|,|a_{ji}|) of integers. For GKM algebras, in addition we have the following: If a_{ii} = 2, i^{th} vertex will be denoted by a white circle and if a_{ii} = 0, i^{th} vertex will be denoted by a crossed circle. If a_{ii} = -k, k > 0, i^{th} vertex will be denoted by a white circle with –k written above the circle within the parenthesis.

Let P^+ ={ λ ε h*/ λ(h_i) ≥0 for all i ε I, λ(h_i) is a positive integer if a_{ii} = 2}. Let V be the irreducible highest weight module over G with the highest weight λ. Let T be the set of all imaginary simple roots counted with multiplicities.

For F ⊂ T, we write F ⊥T if λ(h_i) = 0 for all α_i ε F.

For J ⊂ I^{re}, $= \Delta_J = \Delta \cap (\sum Z\alpha_i), \Delta^{\pm}{}_J = \Delta_J \cap \Delta^{\pm}$,

$$\Delta^{\pm}(J) = \Delta^{\pm} \setminus \Delta_J{}^{\pm}.\ Q_j = Q \cap (\sum Z\alpha_i), Q^{\pm}{}_J = Q_J \cap Q^{\pm}, Q^{\pm}(J) = Q^{\pm} \setminus Q^{\pm}{}_J.$$

Define $g_0^{(J)} = h \oplus (\underset{\alpha \in \Delta_J}{\oplus}), g_{\pm}^{(J)} = \underset{\alpha \in \Delta^{\pm}(J)}{\oplus} g_\alpha$. We obtain the triangular decomposition :

$g = g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}$, where $g_0^{(J)}$ is the Kac Moody algebra associated with the GCM $A_J = (a_{ij})_{i,j \in J}$. $g_-^{(J)}$ and $g_+^{(J)}$ represent the direct sum of irreducible highest weight and lowest weight modules respectively over $g_0^{(J)}$;

W_J = ⟨r_j / jεJ⟩ be the subgroup of W generated by the simple reflections.

Let $W(J) = \{w\ \varepsilon\ W /\ w\Delta^- \cap \Delta^+ \subset \Delta^+(J)\}$.

Proposition 2.3[11,28]:

$$H_k^{(J)} = \underset{\substack{w \in W(J) \\ F \subset T \\ l(w)+|F|=k}}{\oplus} V_J(w(\rho - s(F) - \rho)$$

where $V_J(\mu)$ denotes the irreducible highest weight module over $g_0^{(J)}$ with highest weight μ ; s(F) denotes the sum of elements in F.

Let the homology space

$H^{(J)} = \sum_{\substack{w \in W(J) \\ F \subset T \\ l(w)+|F| \ge 1}} (-1)^{l(w)+|F|+1} V_J(w(\rho - s(F)) - \rho)$; $P(H^{(J)})$={α ε $Q^-(J)$ / dim $H_\alpha \ne 0$}with d(i)=dim $H_{\tau_i}^{(J)}$ for i=1,2,… Let

$T^{(J)}(\tau)$ = {n=$(n_i)_{i \ge 1}$/n_i ε$Z_{\ge 0,}$Σ $n_i\tau_i$= τ}; Let |n|=Σ n_i and define the Witt partition function $W^{(J)}(\tau)$ =

$$\sum_{n \in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod (d(i))^{n_i},\ \ for\ \tau \in Q^-(J).$$

Theorem 2.4[9-11]: Letα ε $\Delta^-(J)$ be a root of a symmetrizable GKM algebra g. Then dim

$$g_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) W^{(J)}(\alpha/d) = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}(\alpha/d)} \frac{(|n|-1)!}{n!} \prod d(i)^{n_i}$$

where μ is the classical Mobius function.

The Kostants formula given in[29] was repeatedly used to determine the root multiplicities in[10,11].

Proposition 2.5[11, 18]: Suppose that a Borcherds-Cartan matrix A=$(a_{ij})_{i,j=1}^{n}$ of charge m = (m_i ε I^{re}/ i ε I} satisfies : i)I^{re} is finite, ii) a_{ij}≠0 for all i,j in I^{im}. Let J = I^{re} and the corresponding decomposition of the GKM algebra is g=g(A, m) $= g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}$.

Then the algebra $g_-^{(J)} = \underset{\alpha\in\Delta^-(J)}{\oplus} g_\alpha$ (respectively, $g_+^{(J)} = \underset{\alpha\in\Delta^+(J)}{\oplus} g_\alpha$) is isomorphic to the free Lie algebra generated by the space $V = \underset{i\in I^{im}}{\oplus} V_J(-\alpha_i)^{\oplus m_i}$ (respectively, $V^* = \underset{i\in I^{im}}{\oplus} V_J^*(-\alpha_i)^{\oplus m_i}$) where $V_J(\mu)$ (resp. $V_{J^*}(\mu)$) denotes the irreducible highest weight (resp. lowest weight) module over the Kac Moody algebra $g_o(J)$ with highest weight μ (resp. lowest weight -μ).

Under these assumptions, the GKM algebra g= $g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}$. is isomorphic to the maximal graded Lie algebra with local part $V \oplus g_0^{(J)} \oplus V^*$.

Theorem 2.6[19,30,31]**:** Let $\wedge(\vee_0)$ be the basic representation of the affine Kac-Moody algebra$A_n^{(1)}$, and let λ be a weight of $\wedge(\vee_0)$. Then, $\dim(V(\Lambda_0))\lambda = p^{(n)}\left(1-\frac{(\lambda,\lambda)}{2}\right)$,where the function $p^{(n)}(m)$ are defined by

$$\sum_{m=0}^{\infty} p^{(n)}(m)q^m = \frac{1}{\phi(q)^n} = \frac{1}{\prod_{j\geq 1}(1-q^j)^n}$$

Remark 2.8: The above formula can be restated as $\dim(\wedge(\vee_0))\lambda = p^{(n)}(m)$, where $\lambda=\wedge 0-m\delta$, $\wedge_0$ is the highest weight, δ is the null root and $m \in Z_+$

Definition 2.9[20]**:**Let $A=(a_{ij})_{i,j=1}^n$, be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S(A) to be of Quasi Affine (QA) type if S(A) has a proper connected sub diagram of affine type with n-1 vertices. The GCM A is of QA type if S(A) is of QA type. We then say the Kac-Moody algebra g(A) is of QA type.

Definition2.10[16]: We say a GGCM $A=(a_{ij})_{i,j=1}^n$ is of Quasi Affine type if A is of indefinite type and the Dynkin diagram associated with A has a connected, proper sub diagram of affine type, whose GCM is of order n-1. We then say the associated Dynkin diagram and the corresponding GKM algebra to be of quasi affine type.

Remark: The GGCM of extended hyperbolic type forms a subclass of quasi affine type and also, not every quasi affine GGCM is of extended hyperbolic type.

3. Classification of Dynkin Diagrams Associated with the Class of $QAGGD_3^{(2)}$

In this section we give the complete classification of connected, non isomorphic Dynkin diagrams associated with the family of GKM algebras $QAGGD_3^{(2)}$.

Theorem 3.1: (Classification Theorem) : The general form of connected non-isomorphic Dynkin diagrams associated GGCM, of quasi affine type $QAGGD_3^{(2)}$ are classified as in Table 1 and any Dynkin diagram of $QAGGD_3^{(2)}$ is one of the types of 1818 Dynkin diagrams listed in Table 1.

Proof: Start with the affine Dynkin diagram $D_3^{(2)}$ of Kac Moody type;

We extend this Dynkin diagram by adding a 4th vertex, which is connected to either one, two or all the three vertices of the affine diagram so that we have the possible Dynkin diagrams and the associated GGCM, of quasi affine type $QAGGD_3^{(2)}$.

Here can represent one of the possible 9 edges, through which the fourth vertex is connected to the affine diagram :

The different exhaustive cases of adding the fourth vertex to the base affine diagram are described in Table 1;

Thus we see that there are 1818 types of connected, non isomorphic Dynkin diagrams associated with the GGCM in the family $QAGGA_2^{(1)}$. The above discussion also proves that any Dynkin diagram associated with the quasi affine Generalized Generalized Kac Moody algebra $QAGGD_3^{(2)}$ will belong to one of the above mentioned 1818 types (by construction).

Root Multiplicities for the GKM algebras $QAGGD_3^{(2)}$ with one Imaginary Simple Root whose Associated GGCM is $\begin{pmatrix} -k & -a & 0 & 0 \\ -a & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$ where k, a are non negative integers.

In this section, we compute the root multiplicities for the GGCM associated with $QAGGD_3^{(2)}$ with one imaginary simple root . Note that the notations given in the earlier section are followed here, for computing the root multiplicities.

Consider the GKM algebra g = g(A, <u>m</u>) associated with the Borcherds-Cartan matrix, whose GGCM is $\begin{pmatrix} -k & -a & 0 & 0 \\ -a & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$

The symmetrizable decomposition for GGCM A is given by $A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1/2&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}-k&-a&0&0\\-a&2&-2&0\\0&-2&4&-2\\0&0&-2&2\end{pmatrix}$.

This GKM algebra is obtained as an extension of $D_3^{(2)}$ of charge $\underline{m}$ = (s,1,1,1), where k,s are non negative integers. Note here that A is a symmetric matrix.

The Index set for the simple roots of g is I = {1,2,3,4}; Imaginary simple root = $\{\alpha_1\}$

Real simple roots = $\{\alpha_2, \alpha_3, \alpha_4\}$; T = $\{\alpha_1, \alpha_1, \ldots, \alpha_1\}$ counted s times.

Since (α_1, α_1) = -k < 0, the set F can be either empty or F = $\{\alpha_1\}$;

W taking J={2,3,4}, $g_0^{(J)} = g_0 \oplus C\, h_1$ where $g_0 = \langle e_2, f_2, e_3, f_3, e_4, f_4\rangle \approx A_2^{(1)}$ and W(J) = {1}.

we have $H_1^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ = (s copies); $H_2^{(J)} = H_3^{(J)} = \ldots = 0$;

Hence $H^{(J)} = V_J(-\alpha_1) \oplus \ldots \oplus V_J(-\alpha_1)$ (s copies) where $V_J(-\alpha_1)$ is the standard representation of $D_3^{(2)}$ with highest weight - α_1. Here, $\wedge_0 = -\alpha_1$;

Let $\lambda = \wedge_0 - m\,\delta$;.

Identifying $-j\alpha_1 - l\,\alpha_2 - m\alpha_3 - n\alpha_4 \in Q^-$ with (j,l,m,n) $\in Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0}$, the weights of $V_J(-\alpha_1)$ are listed as:

$P(H^{(J)})$={(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)} where $\dim H^{(J)}_{(1,0,0,0)} = \dim H^{(J)}_{(1,1,0,0)} = \dim H^{(J)}_{(1,1,1,0)} = \dim H^{(J)}_{(1,1,1,1)}$ = s.

For a weight λ = (1,l,m,n),dim $V_J(-\alpha_1)\lambda = p^{(2)}$(- k – (λ , λ)) / (2a) ;

For λ =(1, l, m, n) = $\alpha_1 + l\alpha_2 + m\alpha_3 + n\alpha_4$, we compute (λ, λ) = -k-2al+2l^2-4ml+4m^2-4mn+2n^2;

(- k –(λ , λ)) / (2a) =1/a {(-al+l^2+m^2-2ml-2mn+n^2};

(We choose a such that 1/a {(-al+l^2+m^2-2ml-2mn+n^2} is an integer)

Hence dim $V_J(-\alpha_1)\lambda$ = $p^{(2)}$ (1/a {(-al+l^2+m^2-2ml-2mn+n^2}) where the function $p^{(2)}$ are defined by

$$\sum_{m=0}^{\infty} p^2(m)q^m = \frac{1}{\phi(q)^2} = \frac{1}{\prod_{j\geq 1}(1-q^j)^2} \qquad \text{Eqn (3.1)}$$

Hence, dim $H_\lambda^{(J)}$ = s . $\mathbf{p}^{(2)}$(- k –(λ , λ)) / (2a) .

We then have $P(H^{(J)}) = \{\tau_i / i \geq 1\}$, where τ_1 = (1,0,0,0), τ_2 =(1,1,0,0), τ_3 = (1,1,0,1), τ_4 = (1,1,1,0), etc.

Svery root of g is of the form (j,l,m,n) for j ≥ 1, l, m, n ≥ 0, the Witt partition function $W^{(J)}(\tau)$ becomes

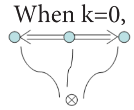
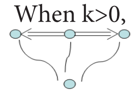
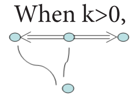
$$W^{(J)}(\tau) = \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod s\, p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2a}\right)^{n_i}.$$

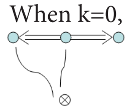
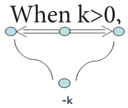
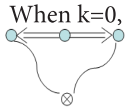
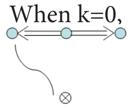
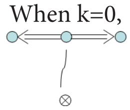
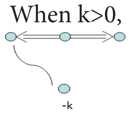
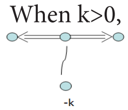
Thus we have proved the following theorem which explicitly gives the root multiplicity formula:

Theorem3.2: Let g=g(A, $\underline{m}$) be the Quasi affine Generalized Generalized Kac Moody algebra associated with the Borcherds Cartan matrix A= $\begin{pmatrix}-k&-a&0&0\\-a&2&-2&0\\0&-1&2&-1\\0&0&-2&2\end{pmatrix}$ of charge $\underline{m}$ = (s,1,1,1) where k,s are non negative integers. Then for any root $\alpha = -k_1\alpha_1 - k_2\alpha_2 - k_3\alpha_3 - k_4\alpha_4$ with k_i's as non negative integers, the root multiplicity of α is given b $\sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod s\, p^{(2)}\left(\frac{-k-(\tau_i,\tau_i)}{2a}\right)^{n_i}$, where μ denotes the Classical Mobius function and the function $p^{(2)}$ are given by equation 3.1 .

Remark 3.3:Here the GKM algebra g is isomorphic to the maximal graded Lie algebra with local part $H^{(J)} \oplus$ ($\mathbf{D_3^{(2)}}$ +h) $\oplus H^{(J)*}$.

Table 1.

Extended Dynkin diagram of quasi affine type QAGGD$_3^{(2)}$	Corresponding GGCM	Number of possible, connected Dynkin diagrams
When k=0, 	$\begin{pmatrix}0&-a_1&-a_2&-a_{-3}\\-b_1&2&-2&0\\-b_2&-1&2&-1\\-b_3&0&-2&2\end{pmatrix}$	729 (9x9x9=729)
When k>0, 	$\begin{pmatrix}-k&-a_1&-a_2&-a_{-3}\\-b_1&2&-2&0\\-b_2&-1&2&-1\\-b_3&0&-2&2\end{pmatrix}$	729 (9x9x9=729)
When k>0, 	$\begin{pmatrix}-k&-a_1&-a_2&0\\-b_1&2&-2&0\\-b_2&-1&2&-1\\0&0&-2&2\end{pmatrix}$	81(9x9=81)

When k=0, 	$\begin{pmatrix} 0 & -a_1 & -a_2 & 0 \\ -b_1 & 2 & -2 & 0 \\ -b_2 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$	81(9x9=81)
When k>0, 	$\begin{pmatrix} -k & -a_1 & 0 & -a_3 \\ -b_1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ -b_3 & 0 & -2 & 2 \end{pmatrix}$	81(9x9=81)
When k=0, 	$\begin{pmatrix} 0 & -a_1 & 0 & -a_3 \\ -b_1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ -b_3 & 0 & -2 & 2 \end{pmatrix}$	81(9x9=81)
When k=0, 	$\begin{pmatrix} 0 & -a & 0 & 0 \\ -b & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$	9
When k=0, 	$\begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & 2 & -2 & 0 \\ -b & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$	9
When k>0, 	$\begin{pmatrix} -k & -a & 0 & 0 \\ -b & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$	9
When k>0, 		9

4. Conclusion

In this work, the class of quasi affine GKM algebras $QAGGD_3^{(2)}$ are defined, the general form of connected, non isomporphic associated Dynkin diagrams are classified; root multiplicities for specific classes in $QAGGD_3^{(2)}$ are computed; There is further scope for understanding the structure and computing the root multiplicities of other quasi affine GKM algebras of indefinite type;

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