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Fuzzy Bayesian Inference for Gompertz Distribution

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Abstract

Objectives: Fuzzy Bayesian approach is implemented to enrich the probability updating process with fuzzy facts. **Methods:** In this paper, different methods of estimation are discussed for the parameters of Gompertz distribution when the available data are in the form of fuzzy numbers. Bayes estimators of the parameters are studied under different symmetric and asymmetric loss functions. The estimation procedures are discussed in details and compared via Monte Carlo simulations. Finally, a real data set which shows the TB affected people of the thirty districts of Tamil Nadu in the year 2009 to 2011 is investigated to explain the applicability of the proposed methods. **Findings:** Among all the loss functions which are provided here, Linear Exponential loss function is more preferable as compared to all other loss functions.

Keywords: Bayesian Estimation, Gompertz Distribution, Loss Functions, Simulation
Mathematical Subject Classification: 628A6

1. Introduction

Recently, in the study of statistical methodology, many authors have studied the characterization of Gompertz distribution. The Gompertz distribution is applied in various fields such as actuarial science, epidemiological and biomedical studies. Many methods have been implemented for estimating the parameters of the Gompertz distribution such as maximum likelihood method[1], least square method[2] etc. Also recently, research studies have been developed in which a combination of Fuzzy Set Theory and Statistics has been established with different purposes. Dual fuzzy systems approach has been already used to approximate nonnegative symmetric solution of some dual fuzzy linear system of equations[3]. To improve the stability function, Hybrid Fuzzy Jordan Network method has been introduced[4]. Refined Asymmetric Classifier Fuzzy Keyword Search is another new scheme used[5]. Fuzzy Bayesian approach is implemented to enrich the probability updating process with fuzzy facts. In this paper, different methods of estimation are discussed to estimate the parameters of Gompertz distribution when the available data are in the form of fuzzy numbers.

2. Gompertz Distribution

A random variable X is said to possess a Gompertz distribution if it has the following form:

$$f(x) = b\eta e^{bx} e^{\eta} e^{-\eta e^{bx}} \text{ where } b > 0 \text{ and } \eta > 0 \tag{1}$$

3. Fuzzy Data and the Likelihood Function

If an evidence e, has n fuzzy values, $e_1, e_2, \ldots, \ldots, e_n$, the likelihood $p(e_i|H_j)$ is computed as

$$p(e_i / H_j) = \int \mu_{e_i}(x) f(x / H_j) dx$$

where $f(x / H_j)$ density function. The posterior probability is:

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$$p(H_j / e_i) = \frac{\int \mu_{e_i}(x) f(x/H_j) dx \; p(H_j)}{\sum_{k=1}^{m} \left[\int \mu_{e_i}(x) f(x/H_j) dx \; p(H_j) \right]}$$

Suppose that $X_1 \ldots X_n$ is a random sample of size 'n' from Gompertz distribution with pdf given by Equation (1). Let X =... $(X_1 \ldots X_n)$denote the corresponding random vector. If a realization $x = (x_1 \ldots x_n)$ of X was known exactly, we can obtain the complete data likelihood function as

$$L(x, b, \eta) = \left(b\eta \, e^{\eta}\right)^n e^{b\sum x_1 - \eta e^{b\sum x_i}}, x, b, \eta > 0$$

Now suppose that x is not observed precisely and only partial information is available in the form of a fuzzy subset $\tilde{x}$ with the membership function $\mu_{\tilde{x}}(x)$ [7]. This information can be encoded as a trapezoidal fuzzy number $\tilde{x}_i$. For trapezoidal membership functions, the trapezoidal fuzzy can be defined as $\tilde{x} = (a, b, c, d)$ and its membership function can be defined as

$$\mu_{\tilde{x}}(\gamma) = \begin{cases} \dfrac{\gamma - a}{b - a}; & a \le \gamma \le b \\ 1 & b \le \gamma \le c, \\ \dfrac{d - \gamma}{d - c}; & c \le \gamma \le d, \\ 0 & otherwise. \end{cases}$$

Assuming the joint membership function $\mu_{\tilde{x}}(x)$, the likelihood can be written as

$$L(b, \eta; \tilde{x}) = \prod_{i=1}^{n} \int f(x; b, \eta) \mu_{\tilde{x}_i}(x) dx$$

4. Bayesian Estimation

In this section, let us assume bivariate prior distribution of the parameters α and β, to obtain the posterior distribution.

The bivariate prior distribution of the Gamma and Log normal priors is given by

$$u(b, \eta) = u_1(b/\eta).u_2(\eta) = \frac{\beta}{\alpha} \frac{\lambda}{} b \quad e \quad \frac{}{\eta\sigma\sqrt{\ \pi}} e^{\frac{(\ln\eta \quad \mu)}{}} \; where,$$

$$u_1(b, \eta) \frac{{}^{\alpha}e^{-\beta} \, b^{\alpha-}}{(\alpha)} \; and \; u_2(\eta) \quad \frac{}{\eta\sigma\sqrt{2\pi}} e^{\frac{(\ln\eta \quad \mu)}{}}$$

Hence the Posterior distribution based on the above priors is

$$\pi^*(b, \eta / \tilde{x}) = \frac{u_1(b/\eta) u_2(\beta) \; L(b, \eta; \tilde{x})}{\iint u_1(b/\eta) u_2(\eta) \; L(b, \eta; \tilde{x}) db d\eta}$$

The Bayesian estimation for the above expression is obtained by different methods of approximation as detailed below.

5. The Loss Functions

In Bayesian outlook the loss functions play an important role. Many authors use the symmetric loss function and obtain the posterior mean as the Bayesian estimate[8]. In this section we use both the types of loss functions to obtain the posterior mean.

5.1 Asymmetric Loss Functions

5.1.1 Linear Exponential (LINEX) Loss Function

In Linear Exponential (LINEX) loss function the Bayes estimator of θ, is given as $\hat{\theta}_L = -\frac{1}{a} \ln E_\theta[\exp(-a\theta)]$ provided that E_θ [exp(-aθ)] exists and is finite[9]. The Bayes estimator is given as

$$\hat{u}_L = E[\exp(-ab), \exp(-a\eta)/y] = \frac{\iint u[\exp(-ab), \exp(-a\eta)] \pi(b, \eta) db d\eta}{\iint \pi(b, \eta) db d\eta} \tag{2}$$

The above integral can be approximated using Linley's approximation procedure

Lindley's expansion can be approximated asymptotically by

$$\hat{\theta} = u + \frac{1}{2}[(u_{11}\delta_{11}) + (u_{22}\delta_{22})] + u_1\rho_1\delta_{11} + u_2\rho_2\delta_{22} + \frac{1}{2}[(L_{30}u_1\delta_{11}^2)(L_{03}u_2\delta_{22}^2)]$$

where L is the log- likelihood function, and the estimated value is given by

$$\hat{\theta} = \frac{\beta^{\alpha} e^{-\beta b} b^{\alpha-1}}{\sqrt{2\pi}\Gamma(\alpha)\eta\sigma} e^{-\frac{(\log\eta - \mu)^2}{2\sigma^2}} + \frac{a^2}{2}\left[\frac{b^2 e^{-ab}}{n - \eta b^2 m^2 e^{bm}} + \frac{\eta^2 e^{-a\eta}}{n}\right] - \left(\frac{abe^{-ab}(-\beta b + \alpha - 1)}{n - \eta b^2 m^2 e^{bm}}\right)$$
$$+ \frac{\eta a e^{-a\eta}}{n}\left(-1 + \frac{1}{2\sigma^2}\right) + \left[\left(2n - \eta b^3 m^3 e^{bm}\right)\left(\frac{1}{n - \eta b^2 m^2 e^{bm}}\right)^2 \left(\frac{a^2 b e^{-a(b+\eta)}\eta}{n}\right)\right]$$

5.1.2 Entropy Loss Function

The Bayes estimator for the Entrophy loss function (ENLF) is

$$\hat{u}_G = E\{u[(b)^{-k},(\eta)^{-k}]/t\} = \frac{\iint u[(b)^{-k},(\eta)^{-k}]\pi(b,\eta)\,db\,d\eta}{\iint \pi(b,\eta)\,db\,d\eta} \tag{3}$$

Similar Lindley approach is used for the general entropy loss function as in the LINEX loss and the estimated value is given by

$$\hat{\theta} = \frac{\beta^{\alpha} e^{-\beta b} b^{\alpha-1}}{\sqrt{2\pi}\Gamma(\alpha)\eta\sigma} e^{-\frac{(\log\eta-\mu)^2}{2\sigma^2}} + \frac{k(k+1)}{2}\left[\frac{b^{-k}}{n-\eta b^2 m^2 e^{bm}} + \frac{\eta^{-k}}{n}\right] - \left(\frac{k b^{-k}(-\beta b+\alpha-1)}{n-\eta b^2 m^2 e^{bm}}\right)$$
$$-\left(-\frac{1}{\eta}+\frac{1}{2\sigma^2\eta}\right)\frac{k\eta^{1-k}}{n} + \left[k^2 b^{-k}\eta^{-k}\left(\frac{2n-\eta b^3 m^3 e^{bm}}{n}\right)\left(\frac{1}{n-\eta b^2 m^2 e^{bm}}\right)^2\right]$$

5.1.3 Squared Error Loss Function

The squared error loss function is symmetric in nature. The Bayes estimator of a function $\hat{u} = u(b,\eta)$ under squared error loss function is the posterior mean.

$$\text{where } \hat{u} = E[u(b,\eta)/t] = \frac{\iint u(b,\eta)\pi^*(b,\eta)\,db\,d\eta}{\iint \pi^*(b,\eta)\,db\,d\eta} \tag{4}$$

The same Lindley approach is used to find the estimated value.

Also to evaluate the integrals in Equation (4) for SELF, the Tierney and Kadane (TK) estimation[10] is used in the following form:

$$\hat{g}(b,\eta) = \left[\frac{\det\sum}{\det\sum^*}\right]^{\frac{1}{2}} \exp\left\{n\left[H^*(b,\eta) - H(b,\eta)\right]\right\}$$

$$H(b,\eta) = \frac{1}{n}\left[\begin{array}{l} -n\log\Gamma(b) + n\log\eta^{\beta} - \left(\frac{(b-\beta)^2}{2\alpha^2}\right) - \frac{\sigma}{\beta} - \log\beta^{\mu+1} \\ +\sum_{i=1}^{n}\log\int x^{b-1}e^{-n\bar{x}\beta}\mu_{\tilde{x}_i}(x)dx \end{array}\right]$$

$$H^*(b,\eta) = -\frac{1}{n}\left(\log b + \log\sqrt{2\pi} + \frac{(b-\beta)^2}{2\alpha^2} - \mu\log\sigma + \log\Gamma(\mu) + (\mu+1)\log\beta + \frac{\sigma}{\beta}\right) +$$
$$\frac{1}{n}\left[-n\log\Gamma(\alpha) - n\alpha\log\beta - \left(\frac{(b-\beta)^2}{2a^2}\right) - \frac{\sigma}{\beta} - \log\eta^{\mu+1} + \sum_{i=1}^{n}\log\int x^{\alpha-1}e^{\frac{-n\bar{x}}{\beta}}\mu_{\tilde{x}_i}(x)dx\right]$$

$$H_1^* = \frac{-1}{n}\left[\frac{(b-\beta)}{a^2}\right] + H_1, H_{11}^* = \frac{-1}{n}\left[\frac{1}{\alpha^2}\right] + H_{11}, H_2^* = \frac{-1}{n}\left[\frac{(\mu+1)}{\eta} - \frac{1}{\eta^2}\right] + H_2$$

$$H_{22}^* = \frac{-1}{n}\left[\frac{-(\mu+1)}{\eta^2} + \frac{2}{\eta^3}\right] + H_{22}, H_{12}^* = 0$$

$$\det\sum = (H_{11}H_{22} - H_{12}^2)^{-1}, \det\sum^* = (H_{11}^*H_{22}^* - (H_{12}^*)^2)^{-1}$$

6. Numerical Data Analysis

To exemplify the application of the methods of estimation developed above we consider the set of data reported by NTBC from 2009 to 2011. The report provides district wise suspected TB cases and the smear positive TB cases. In Tamil Nadu state of India there are 30 states. The TB patients of each state from the year 2009 to 2011 are taken for the study. Each realization of x for the years 2009, 2010 and 2011 was made fuzzy using the following membership functions.

$$\mu_{A_{2009}}(\gamma) = \begin{cases} \dfrac{\gamma}{\delta_0} & \gamma_0 - 3\delta_0 \le \gamma < \gamma_0 + \delta_0 \\ -\left(\dfrac{\gamma}{\delta_0} + 40\delta_0\right) & \gamma_0 + \delta_0 \le \gamma < \gamma_0 + 2\delta_0 \\ \dfrac{\gamma}{\delta_0} - 40\delta_0 & \gamma_0 + 2\delta_0 \le \gamma \le \gamma_0 + 3\delta_0 \\ 0 & otherwise \end{cases} \tag{5}$$

$$\mu_{A_{2010}}(\gamma) = \begin{cases} \dfrac{\gamma}{\delta_0} & \gamma_0 - 3\delta_0 \le \gamma < \gamma_0 + \delta_0 \\ \left(\dfrac{\gamma}{\delta_0} - 20\delta_0\right) & \gamma_0 + \delta_0 \le \gamma < \gamma_0 + 2\delta_0 \\ \dfrac{-\gamma}{\delta_0} + 50\delta_0 & \gamma_0 + 2\delta_0 \le \gamma \le \gamma_0 + 3\delta_0 \\ 0 & otherwise \end{cases} \tag{6}$$

$$\mu_{A_{2011}}(\gamma) = \begin{cases} \dfrac{\gamma}{\delta_0} & \gamma_0 - 3\delta_0 \le \gamma < \gamma_0 + \delta_0 \\ \left(\dfrac{\gamma}{\delta_0} - 10\delta_0\right) & \gamma_0 + \delta_0 \le \gamma < \gamma_0 + 2\delta_0 \\ \dfrac{-\gamma}{\delta_0} + 50\delta_0 & \gamma_0 + 2\delta_0 \le \gamma \le \gamma_0 + 3\delta_0 \\ 0 & otherwise \end{cases} \tag{7}$$

In analysing the complete data and applying the different methods as given above, the estimates using the maximum likelihood estimation and Bayes' estimators of the parameters of the Gompertz model are studied for different loss functions under bivariate priors. From Table 1, the loss function LINEX loss function has lesser Bayes posterior risk than other loss functions and can be considered as the better loss function than the other loss functions.

7. Simulation Study

Simulation is employed to examine the performance of a different field of study. Here, a simulation criterion is used and the Bayes estimates are calculated under different loss functions along with the bivariate gamma and log-normal prior. The simulation was carried out for sample sizes$n$ = 50, 75, 100, 200 and 500. The comparison of Bayes posterior risk under different loss functions the bivariate prior has been made. From Table 2 we can conclude that LINEX loss function is more preferable as compared to all other loss functions which are provided here because under this loss function Bayes posterior risk is small.

8. Conclusion

In this study 'Fuzzy Bayesian Inference for Gompertz Distribution', different estimation procedures for the Gompertz distribution were obtained when the data obtained are fuzzy numbers. Different types of loss functions both symmetric and asymmetric were used for estimating the posterior risk for the data of TB affected people of Tamil Nadu state. Amongst loss functions, LINEX loss function is more preferable as compared to all other loss functions which are provided here because under this loss function Bayes posterior risk is small.

Table 1. Posterior summary using the real data

YEAR	α	β	μ	σ	SELF		LINEX	ELF
					LA	TK		
2009	0.291	0.4768	0.5	0.6	0.712	0.935	0.232	0.345
2010	0.2496	0.29792	0.5	0.6	0.365	0.964	0.146	0.291
2011	0.2251	0.4381	0.5	0.6	0.765	0.9824	0.540	0.421

Table 2. Bayes estimates under different loss functions for the years 2009, 2010 and 2011

Year	n	SELF		LINEX	ELF
		LA	TK		
2009	50	0.595	0.892	0.212	0.452
	75	0.452	0.855	0.247	0.245
	100	0.419	0.851	0.211	0.237
	200	0.392	0.752	0.204	0.403
	500	0.388	0.754	0.192	0.543
2010	50	0.781	0.918	0.336	0.240
	75	0.683	0.927	0.294	0.354
	100	0.532	0.819	0.213	0.750
	200	0.511	0.800	0.211	0.564
	500	0.473	0.775	0.193	0.245
2011	50	0.517	0.781	0.249	0.173
	75	0.428	0.683	0.221	0.547
	100	0.417	0.532	0.189	0.553
	200	0.298	0.511	0.184	0.521
	500	0.161	0.473	0.179	0.594

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