Indian Journal of Science and Technology, Vol 9(7), DOI: 10.17485/ijst/2016/v9i7/87849, February 2016
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Spectral Properties of Differential Operators Constructed of Sectorial Forms

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Abstract

Objectives: Spectral properties and asymptotic distribution of eigenvalues of differential operators has been an important subject in mathematics and physics because they reflect many properties of the operators, moreover we can calculate some equalities and inequalities about differential operators. **Methods:** In this paper the main concepts and the basics of sectorial forms and m-sectorial operators are discussed in detail. Then we introduce a sectorial form that has an integral form and conclude some properties about it. Finally some spectral properties of differential operators constructed of this proposed sectorial form are studied. **Finding:** It is usual to prove spectral properties of differential operators by investigating resolvent estimates but in this paper we prove an important spectral theorem by using properties of bilinear forms instead of resolvent estimates. **Improvement:** We improve the method of proving the spectral theorems by using sectorial forms and getting a useful theorem about spectral properties of differential operators.

Keywords: Asymptotic Distribution, Eigenvalues,m-Sectorial Operators, Sectorial Forms, Spectral Properties

1. Introduction

[1]Showed that the closed sectorial forms are suitable means to construct the m-sectorial operators and introduced some m-sectorial operators of sectorial forms. On the other hand finding the spectral properties and asymptotic distribution of differential operator's eigenvalues are very important in mathematics, physics and engineering[2]. So the spectral properties of differential operators have been investigated by some authors[3–10]. In this paper the proposed form:

$$t[u,v]=\int_0^1 h(t)\mu(t)u`(t)\overline{v`(t)}dt$$

is considered and some spectral properties will be proved about the differential operator T associated to form t.

In this paper at the first step an introduction is presented. Then, we introduce some preliminaries, some basic definitions and theorems about sectorial forms and m-sectorial operators. In Section 3 we conclude some properties about the proposed form t and finally we prove an important spectral theorem that states asymptotic distribution of eigenvalues of the m-sectorial operator T constructed of the sectorial form t.

2. Preliminaries

We start by introducing some basic terms and results from the theory of sectorial forms. We consider the sesquilinear form a defined on a subspace *L* of a separable Hilbert space *H*. The sesquilinear form *a: LxL→C* is given such that *a[u,v](u∈L,v∈L)* is complex valued and varies linearly in u∈L for each constant v∈L and semi-linear in *v∈L* for each fixed *u∈L*. Here L is named the domain of a and is denoted by *D(a)* and is obtusely defined in *H*.

Definition 2.1

A form a is considered to be symmetric while:

$$a[u,v]=\overline{a[u,v]}(u,v\in D(a)).$$

With each form an associate another form a^* by;

$$a^*[u,v]=\overline{[v,u]}(D(a)=D^*(a)).$$

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a^* is named the adjoint of the form a

Definition 2.2

A form a is symmetric if and only if $a=a^*$.

Definition 2.3

The set of values of the form a i.e.

$$\Theta(a)=\{a[u,u];u\in D(a)=L;\|u\|=1\}$$

is called the numerical range of a.

Definition 2.4

The form a is called to be sectorial if $\Theta(a)$ is a subset of a sector of the form:

$$S=\{z\in C:\left|\arg(z-\gamma)\right|\leq\theta\}.0\leq\theta<\frac{\pi}{2},\gamma\in R\}.$$

Here γ and ɵ are a vertex and a semi-angle of the form a respectively.

Definition 2.5

A form a is called a closed form if $D[a]$ is complete with respect to the following norm:

$$\|u\|=\left(\mathrm{Re}(a[u,u])+\frac{\delta}{M}\|u\|^2\right)^{\frac{1}{2}}. \quad (2.1)$$

Definition 2.6

An operator T in H is considered to be accretive if the numerical range of T which will be denoted by $\Theta(T)=(Tu,v)$, is a subset of the right half-plane, that is if: $Re(Tu,v)\geq 0$ *for all* $u\in D(T)$.

Definition 2.7

We call an operator T in H to be m-accretive if for $\mathrm{Re}\lambda>0,\|T+\lambda I\|\leq(\mathrm{Re}\lambda)^{-1}$

Definition 2.8

We call an operator T in H to be sectorial operator if the numerical range $\Theta(T)$, is not only a subset of the right half-plane, but also is a subset of a sector of the form:

$$S=\{z\in C:\left|\arg(z-\gamma)\right|\leq\theta\},0\leq\theta<\frac{\pi}{2},\gamma\in R.$$

Here γ and ɵ are a vertex and a semi-angle of the sectorial operator T respectively.

Definition 2.9

We call an operator T in H to be m-sectorial operator if it is sectorial and m-accretive operator.

Now from[7], some known results and theorems specially First and Second Representation theorems that are about sectorial forms and m-sectorial operators are described as follows:

Theorem 2.1

If $a[u, v]$ is limited to defined on H, then there is a bounded operator Tɵ$B(H)$ such that $a[u, v]=(Tu, v)$.

Theorem 2.2

(First representation theorem) Let the form $a[u,v]$ be an obtusely defined closed, sectorial and sesquilinear form in H. There is an m-sectorial operator T in which:

- $D(T)\subset D(a)$and $a[u,v]=(Tu,v)$ for every $u\in D(T)$and $v\in D(a)$;
- D(T) is considered as a core of a;
- if u ∈D(a)&and w∈H&and a[u,v]remains for every v depended on to a core of a, then u∈D(T) and Tu-w moreover the elements of u∈D(T) satisfy $in\left|a[u,v]\right|\leq M_u\|v\|, forv\in D(a)$.

Let a be a obtusely express, bounded and symmetric form that is ranged from below and let T be the related to self-adjoint operator. The equation $[u,v]=[Tu,v]$ connecting the form a with the operator T is undesirable indeed it is not correct for all $u,v\in D(a)$ since $D(T)$is a proper subset of $D(a)$. A more complete representation of is equipped by the Theorem 2.3.

Theorem 2.3

Let a be a densely expressed, bounded and symmetric form, such thata ≥0, and let $H=T_a$ be the related to self-adjoint operator, then there is a $D(H^{\frac{1}{2}})=D(a)$, so:

$$a[u,v]=(H^{\frac{1}{2}}u,H^{\frac{1}{2}}v),u,v\in=D(a).$$

Theorem 2.4

Let T be an m-sectorial operator with vertex 0 and semi-angle ɵ, H=ReT then is non-negative and there is a symmetric operator $B \in B(H)$, where $\|B\| \le \tan\theta\, so$:

$$T = G(1+iB)G, G = H^{\frac{1}{2}}.$$

3. Results

In this paper we define a proposed form:

$$t[u,v] = \int_0^1 h(t)\pi(t)u'(t)\overline{v'(t)}dt, (3.1)$$

Where h(t)∈C[0,1] and μ(t)∈C^∞[0,1] is a complex function satisfying the following conditions:

$$\mu(t) \ne 0, |\arg\mu(t)| \le \theta, (0 < \theta < \frac{\pi}{2}), (3.2)$$

Also we assume $D(t) = W^0{}_{2,h}(0,1) = \overline{C_0^\infty(0,1)}$ that the closure is considered that:

$$|u|_+ = \left(\int_0^1 h(t)\left|\frac{du(t)}{dt}\right|^2 dt + \int_0^1 |u(t)|^2 dt\right)^{\frac{1}{2}}. (3.3)$$

Theorem 3.1

The form tis expressed by (3.1) is a closed form.

Proof

If in the relation (2.1) instead of the form *a* we set the form t of relation (3.1), then the norm of (3.3) is equivalent to the norm (2.1) and since the space D(t) is a Banach space with respect to norm $|.|_+$ so D(t) sois a Banach space with respect to norm (2.1) therefore by the definition (2.5) the form t is a closed form.

Theorem 3.2

The form t defined by (3.1) is sectorial.

Proof. From the relations (3.2) it can be found that the form t is sectorial.

Theorem 3.3

There exists an m-sectorial operator *T* associate to the form t.

Proof

Since the form $t[u,v]$ satisfies the conditions in Theorem 2.2, then there is an m-sectorial operator *T* while as:

$$D(T) \subset D(t) = W^0{}_{2,h}(0,1)$$

and

$$(Tu,v) \quad \int h(t) \quad (t)u'(t)\overline{v'(t)}dt$$

where.$u \in D(T)$, $v \in D(t)$.

The m-sectorial operator T satisfying the above conditions is unique.

Theorem 3.4

D(T) is the group of all the functions $u(t) (0<t<1)$ that satisfies in $Tu = -(h\mu(t)u'(t))' \in L^2(0,1)$. .

Proof

According to Theorem 2.2:

$$D(T) = \{u \in W^0{}_{2,h}(0,1) : |t[u,v]| \le M_u\|v\|, forv \in W^0{}_{2,h}(0,1)\}$$

Now for u∈D(T) if we let f=Tu then,

.

$$(f,v) = \int_0^1 h\mu(t)u'(t)\overline{v'(t)dt}u \in D(T), v \in C_0^\infty(0,1).$$

$$So Tu = -(h\mu(t)u'(t))' \in L^2(0,1)$$

Theorem 3.5

If T is the m-sectorial operator related to form t in (3.1). Then for $0 \le \theta < \frac{\pi}{2}, k > 0$ there exists such that:

$$N(k) = card\{j : |\lambda_j(T)| \le k, |\arg\lambda_j(T)| \le \theta\} \le M(1+k)^{\frac{1}{2}}.$$

Proof

For every form t[u,v] we denote the real part of t' with T and we define it as:

$$t'[u,v] = \int_0^1 h\mu_0(t)u'(t)\overline{v'(t)}dt. (3.4)$$

Where.$\mu_0(t)$=Re(μ(t)). According to second representation theorem there exist the operators T',S'≥0 such that t(u,v)=T'u,v and $T'=S'^2$. For the form and the number λ≥ the form t_λ is defined by:

$$t_\lambda[u,v] = t[u,v] + \lambda(u,v), D(t_\lambda) = D(t).$$

Suppose t and t be the forms (3.1),(3.4) then the forms t_λ and t'_λ induce m-sectorial operators T_λ and $T_\lambda \ge 0$ it is obvious that:

$$T_\lambda = T + \lambda I, T'_\lambda = T + \lambda I.$$

From Theorem 2.4 it was found that:

$$(T+\lambda I) = (T'+\lambda I)^{\frac{1}{2}}(I+iB(\lambda))(T'+\lambda)^{\frac{1}{2}}, \lambda \geq 0(3.5).$$

Since B(λ)= B(λ*) is a bounded operator, for every u∈L^2(0,1) we will have:

$$\|(I+iB(\lambda))\|^2 = \|u\|^2 + \|B(\lambda)\|^2 \geq \|u\|^2$$

i.e.,

$$\|(I+iB(\lambda))\|^{-1} \leq 1.$$

From here and (3.5) we will have:

$$(3.6)\quad (T+\lambda I)^{-1} = (T'+\lambda I)^{\frac{-1}{2}} X(\lambda)(T+\lambda I)^{\frac{1}{2}}, \|X(\lambda)^*\| \leq 1, \lambda > 0.$$

Similarly, we will have:

$$(T^*+\lambda I)^{-1} = (T'+\lambda I)^{-\frac{1}{2}} X(\lambda)^*(T'+\lambda I)^{\frac{1}{2}}, \|X(\lambda)^*\| \leq 1, \lambda > 0.$$

So it was founded that the operator (T+λI)$^{-1}$ is compact, then the operator T is a countable spectrum and the eigenvalues of the operator T+λI)$^{-1}$ denoted as:

$$(\lambda_1(T)+\lambda)^{-1}, (\lambda_2(T)+\lambda I)^{-1}, \ldots$$

Also we have:

$$\sum_{i=1}^{\infty} |(\lambda_i(T)+\lambda)^{-1}| \leq |T+\lambda I)^{-1}|_1 \leq |(T'+\lambda I)^{-1}|_2^2$$

Here $|.|_2$ is Hilbert Schmidt norm.

Since for each u∈D(T),|arg(Tu,u)|≤∈, then |argλi(T)|≤ɵ, i=1,2,...i.e.,

$$(|\lambda_i(T)|+\lambda)^{-1} \leq M_\theta \,|(|\lambda_i(T)|+\lambda)^{-1}|.$$

$$So\,n(k) = card\{j : (|\lambda_i(T')| \leq k\} \leq M(1+k)^{\frac{1}{2}}, k > 0$$

From the above relations we will have:

$$N(k) = \int_0^k dN(s) \leq 2k \int_0^k (s+\lambda)^{-1} dN(s)$$

$$\leq 2k \int_0^\infty (s+\lambda)^{-1} dN(s) = 2k \sum_{i=1}^{\infty} (\lambda_i(T)+t)^{-1} \leq 2kM_\theta . |(G+kI)^{-1}|_2^2 .$$

On the other hand:

$$|T'+kI)^{-1}|_2^2 = \sum_{i=1}^{\infty} (\lambda_i((T'+k)^{-1})^2 \sum_{i=1}^{\infty} |(\lambda_i(T')+k)^{-2}|$$

$$= \int_0^\infty \frac{dn(s)}{(k+s)^2} = 2\int_0^\infty \frac{n(s)}{(k+s)^3} = ds$$

$$2\int_0^\infty \frac{n(s)ds}{(k+s)^3} \leq 2\int_0^\infty \frac{1+s)^{\frac{1}{2}} ds}{(k+s)^3} \leq 2M.(1+k)^{\frac{-3}{2}} = ds = ds$$

so

$$N(k) = card\{j : |\lambda_j(T)| \leq k, |\arg \lambda_j(T)| \leq \theta\} \leq M(1+k)^{\frac{1}{2}}.$$

4. Conclusion

This paper has discussed on the differential m-sectorial operators that are constructed of sectorial forms to find out a new method in proving spectral theorems about differential operators. In fact by using of the First and Second representation theorems we make a connection between sectorial forms and m-sectorial operators and by using of properties of sectorial forms we get results about distribution of differential operator's eigenvalues.

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