

Topics on the Degenerate Elliptic Differential Operators Far from Self Adjoint Ones

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Abstract

Background: Elliptic differential operators and asymptotic distribution of eigenvalues of them are discussed in many works. **Methods:** In this paper we get some new results about an important differential operator on a Hilbert space. Also asymptotic distribution of eigenvalues of this kind of differential operators are proved with new methods that estimation of the resolvent of the operators is used in this paper. **Finding:** We get some new theorems about the differential operator A . We consider a bounded domain Ω with smooth boundary in \mathbb{R}^n define a norm and find the asymptotic distribution of eigenvalues of the operator $(Au)(x) = -\sum_{i,j=1}^n \left(p^{2a}(x) a_{ij}(x) q(x) u'_{x_i}(x) \right)'_{x_j}$ in the space $H_\ell = L^2(\Omega)^\ell$. **Improvement:** We improve the method of proving this kind of theorems.

Keywords: Asymptotic Distribution, Eigenvalues, Non-Self Adjoint Elliptic Differential Operator, Resolvent

1. Introduction

Let Ω be a bounded domain with smooth boundary in \mathbb{R}^n i.e., $\partial\Omega \in C^\infty$. We introduced the space $H_\ell = W_2^{2a}(\Omega) \times \dots \times W_2^{2a}(\Omega)$ (ℓ -times) as the space of vector functions $u(x) = (u_1(x), \dots, u_\ell(x))$ defined on Ω with finite norm¹⁻⁵:

$$\|u, W_2^{2a}(\Omega)\| = \left(\sum_{i=1}^n \int \rho^{2a}(x) |u'_{x_i}(x)|_{C^\ell}^2 dx + \int_\Omega |u(x)|_{C^\ell}^2 \right)^{1/2}$$

By H_ℓ we denote the closure of $C_0^\infty(\Omega)^\ell$ in H_ℓ , for $\ell=1$ we set $H=H_1$ and $H=H_1$. In this article, we investigate the asymptotic formula for distribution of the eigenvalues (e.v.) of a non-self adjoint elliptic differential operator A defined by:

$$(Au)(x) = -\sum_{i,j=1}^n \left(\rho^{2a}(x) a_{ij}(x) q(x) u'_{x_i}(x) \right)'_{x_j}$$

in space $H_\ell = L_2(\Omega)^\ell$ here,

$$\rho(x) = \text{dist}\{x, \partial\Omega\}, \alpha \in [1, 0], a_{ij}(x) = \overline{a_{ij}(x)}(i, j = 1, 2, \dots, n), \\ a_{ij}(x) \in C^2(\overline{\Omega})(i, j = 1, 2, \dots, n), q(x) \in C^2(\overline{\Omega}, \text{End } C^\ell),$$

Further more assume that for $\forall x \in \overline{\Omega}$, the matrix function $Q(x)$ has simple eigenvalues $\mu_1(x), \dots, \mu_\ell(x)$ arranged in the complex plane in the following way:

$$\arg \mu_j(x) = 0 (1, 2, \dots, v), \mu_j(x) \notin \overline{\phi} (j = v+1, \dots, \ell),$$

Where $\phi = \{z \in \mathbb{C} : |\arg z| < \phi\}, \phi \in (0, \pi)$. We also assume that the matrix function $a_{ij}(x)$ satisfies the uniformly elliptic condition: i.e., there exist $M > 0$ such that for every $S = (s_1, \dots, s_n) \in C^n, x \in \Omega$ we have:

$$|s|^2 \leq M \sum_{i,j=1}^n a_{ij}(x) s_i s_j.$$

Now for a closed extension of the operator A we need to extend its domain to the:

$$D(A) = \left\{ u \in H_\ell \cap W_2^{2a}(\Omega)^\ell : \sum_{i,j=1}^n (\rho^{2a} a_{ij} u_{xi})_{xi} \in H_\ell \right\},$$

$$\text{Where } W_2^{2a}(\Omega) = \left\{ u : \sum_{i=1}^n \int_\Omega |u^{(i)}(x)|^2 dx < \infty \right\}$$

Here and in the sequel the value of the function $z \in (-\pi, \pi]$ and $\|T\|$ denotes the norm of the bounded operator $T : H_\ell \rightarrow H_\ell$.

2. Resolvent Estimate

Theorem 2.1. Let $S \subset \phi \setminus R_+$ be some closed sector with vertex at 0 and as in Section 1, we recall that for $\forall x \in \overline{\Omega}$, the matrix function $q(x)$ has simple eigenvalues $\mu_1(x), \dots, \mu_\ell(x)$ which are different from zero $\arg \mu_j(x) = 0$ for $j = 1, 2, \dots, v$ and

$\mu_1 \notin \bar{S}$ for $j = v+1, \dots, \ell$ Where $\phi\{z \in \mathbb{C} : |\arg z| < \phi\}, \phi \in (0, \pi)$ (i.e., the eigenvalues $\mu_1(x), \dots, \mu_v(x)$ lie on the positive real line inside the angle Φ , and the rest of the eigenvalues $\mu_{v+1}(x), \dots, \mu_\ell(x)$ lie outside of the angle Φ in view of $S \subset \phi \setminus R_+$ implies that all the eigenvalues $\mu_1(x), \dots, \mu_\ell(x)$ lie on the complex plane and outside of the closed sector ϕ . Then the operator has a discrete spectrum and for sufficiently large in modules $\lambda \in S$ the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous and the following estimate is valid:

$$\|(A - \lambda I)^{-1}\| \leq M_S |\lambda|^{-1} \quad (\lambda \in S, |\lambda| \geq C_S),$$

where M_S, C_S are sufficiently large numbers. The symbol $\|\cdot\|$ stands for the norm of a bounded operator in $HorH_\ell$.

Proof of Spectral Theorem 2.1. Base on the assumptions, the eigenvalues of the matrix function have $q(x)$ the following conditions⁹⁻¹¹:

$\mu_k(x) \in C^2(\bar{\Omega}), k = 1, \dots, \ell$ such that $\mu_1(x), \dots, \mu_\ell(x)$ lie on the positive

real line inside the angle Φ and the rest of the eigenvalues $\mu_{v+1}(x), \dots, \mu_\ell(x)$ lie outside of the angle Φ , then in view of the conditions on the eigenvalues, we defined the operators P_k ($k=1, 2, \dots$, on $H=H'$ by

$$(p_{ky})(x) = - \sum_{i,j=1}^n (\rho^{2a}(x) a_{ij}(x) \mu_k y_{xi}(x))_{x_j}$$

$$D(P_k) = \left\{ y \in H \cap W_2^2, loc(\Omega) : \sum_{i,j=1}^n (\rho^{2a} a_{ij} \mu_k y_{x_i})_{x_j} \in H \right\},$$

Since the eigenvalues $\mu_{v+1}(x), \dots, \mu_\ell(x)$ lie outside of the angle Φ therefore for $k \in \{v+1, \dots, \ell\}$, we assume also that there exist distinct numbers: $\psi_{v+1}, \dots, \psi_\ell \in (\phi, \pi) \cup [-\pi, -\phi)$, such that $\arg \mu_j(x) \equiv \psi_j$ ($j = v+1, \dots, \ell$). such that the eigenvalues $\mu_j(x)$ lie on the fix rays in the complex plane which the angle between $(\bar{\Omega})$ each two consequent ray are equals to $\frac{\pi}{16}$.

Thus, by this assumption, we construct the nonnegative functions, $\phi_{k_1}(x), \dots, \phi_m(x) \in C^\infty$ with the following properties:

$$\sum_{i=1}^m \phi_{k_i}^2(x) \equiv 1 (x \in \bar{\Omega}), \left| \arg \left\{ \mu_k(x_1) \mu_k^{-1}(x_2) \right\} \right| < \frac{\pi}{16} \quad (\forall x_1, x_2 \in \sup p \phi_{k_j}),$$

for any $j \in \{1, \dots, m\}$. Let us construct the functions $\mu_{kr}(x) \in C^2(\bar{\Omega})$ satisfying these conditions:

$$\mu_{kr}(x) = \mu_k(x), (\forall x \in \sup p \phi_{kr}), \mu_{kr}(x) \notin \phi, (\forall x \in (\bar{\Omega})),$$

Fork $v+1, \dots, \ell$ and $r = 1, \dots, m$ we have

$$\left| \arg \left\{ \mu_{kr}(x_1) \mu_{kr}^{-1}(x_2) \right\} \right| < \frac{\pi}{16} \quad (\forall x_1, x_2 \in \phi)$$

And there exist numbers, $|Z_{kr}| \in \mathbb{C}$ such that $|Z_{kr}| = 1$ for $\forall x \in \bar{\Omega}, \lambda \in \bar{\Phi}$ for we have

$$C' \leq \operatorname{Re} \{ Z_{kr} \mu_{kr}(x) \}, C' |\lambda| \leq -\operatorname{Re} \{ Z_{kr} \lambda \}, C' > 0.$$

For $k=v+1, \dots, \ell$ by applying the functions $\mu_{kr}(x)$ as the operators P_k on $H = H_1$ above, we defined the operators, P_{kr} , on $H=H_1$ by:

$$(P_{ky})(x) = - \sum_{i,j=1}^n (\rho^{2a}(x) a_{ij}(x) \mu_k y_{xi}(x))_{x_j}$$

$$D(P_k) = \left\{ y \in H \cap W_2^2, loc(\Omega) : \sum_{i,j=1}^n \rho^{2a} a_{ij} \mu_k y_{x_i} \right\}_{x_j} \in H \Big\}.$$

Take $I_r(y) = \sum_j \int_\Omega \rho^{2a}(x) |y_{x_j}(x)|^2 dx$, then from uniformly elliptic condition we have for $c > 0$:

$$c \sum_{i=1}^n |y_{x_i}(x)|^2 \leq \sum_{i,j=1}^n a_{ij}(x) y_{x_i}(x) \overline{y_{x_j}(x)} = \alpha(x) \nabla_x y(x) > C^n, p^\alpha y_{x_j}$$

Then for $y \in D(P_{kr}), k=v+1, \dots, \ell$ according to we have:

$$I(y) \leq M_1 \operatorname{Re} \left\{ Z_{kr} \sum_{i,j=1}^n (\rho^\alpha a_{ij} \mu_{kr} y_{x_j}, \rho^\alpha y_{x_j}) \right\} = M_1 \operatorname{Re} \{ Z_{kr} (P_{kr} y, y) \}.$$

here and in sequel, the symbol (\cdot) denotes the inner product in H . Now according to (2.2) for $\lambda \in \bar{\Phi}$ we have:

$$I(y) + |\lambda| \|y\|_H^2 \leq M_2 \operatorname{Re} \left\{ (Z_{kr} P_{kr} - \lambda Z_{kr} I) y, y \right\} \leq M_2 \|y\|_H \|(P_{kr} - \lambda I) y\|_H,$$

$$I(y) \leq M_2 \|y\|_H \|(P_{kr} - \lambda I) y\|_H.$$

in put $y = (P_{kr} - \lambda I)^{-1} f \in H$ then we have:

$$|\lambda| \|(P_{kr} - \lambda I)^{-1} f\| \leq M_2 \|(P_{kr} - \lambda I)(P_{kr} - \lambda I)^{-1} f\|_H$$

therefore

$$\|(P_{kr} - \lambda I)^{-1} f\| \leq M |\lambda|^{-1} \|f\|_H$$

$$\text{hence } \|(P_{kr} - \lambda I)^{-1}\| \leq M |\lambda|^{-1}$$

$$\|(P_{kr} - \lambda I)^{-1}\| \leq M (\lambda \in \bar{\Phi}, \lambda \neq 0)$$

In (2.3) we put $y = (P_{kr} - \lambda I)^{-1} f, f \in H$, then we will have:

$$\left| \rho^\alpha \frac{\partial}{\partial x_i} (P_{kr} - \lambda I)^{-1} f \right|_H^2 \leq M \|f\|_H \|(P_{kr} - \lambda I)^{-1} f\|_H \leq M |\lambda|^{-1} \|f\|_H^2, (i=1, \dots, n)$$

for proof (2.5) we have:

$$\left| \rho^\alpha \frac{\partial}{\partial x_i} (P_{kr} - \lambda I)^{-1} f \right|_H^2 = \int_\Omega \rho^{2a} \left| \frac{\partial}{\partial x_i} (P_{kr} - \lambda I)^{-1} f \right|^2 dx$$

$$\leq \sum_{i=1}^n \int_\Omega \rho^{2a} \left| \frac{\partial}{\partial x_i} (P_{kr} - \lambda I)^{-1} f \right|^2 dx$$

$$= I(P_{kr} - \lambda I)^{-1} f;$$

i.e. in $I(y)$ above we put $y = (P_{k_r} - \lambda I)^{-1} f$; so that we will have:

$$\left| \rho^\alpha \frac{\partial}{\partial x_i} (P_{k_r} - \lambda I)^{-1} f \right|_H^2 \leq I \left((P_{k_r} - \lambda I)^{-1} f \right).$$

Since by we have $I(y) \leq M_2 |y|_H |P_{k_r} y|_H$ and by (2.4) $\|P_{k_r} (P_{k_r} - \lambda I)^{-1}\| \leq$ therefore if in $I(y) y = (P_{k_r} - \lambda I)^{-1} f$ we set then we will have:

$$\begin{aligned} \left| \rho^\alpha \frac{\partial}{\partial x_i} (p_{k_r} - \lambda I)^{-1} f \right|_H^2 &\leq \left((p_{k_r} - \lambda I)^{-1} f \right) \leq M_2 \left\| (p_{k_r} - \lambda I)^{-1} \right\| \left\| p_{k_r} - \lambda I)^{-1} f \right\| \\ &\leq M_2 M \left\| (p_{k_r} - \lambda I)^{-1} f \right\| \left\| f \right\| \leq M_3 \left\| (p_{k_r} - \lambda I)^{-1} f \right\| \left\| f \right\| \\ &\leq M_3 \left\| (p_{k_r} - \lambda I)^{-1} f \right\| \left\| f \right\|_H^2 \\ \text{i.e. } \left| \rho^\alpha \frac{\partial}{\partial x_i} (p_{k_r} - \lambda I)^{-1} f \right|_H^2 &\leq M_3 M |\lambda|^{-1} \left\| f \right\|_H^2. \text{ that proof, i.e.} \\ \text{implies} \end{aligned}$$

$$\begin{aligned} \left| \rho^\alpha \frac{\partial}{\partial x_i} (p_{k_r} - \lambda I)^{-1} f \right|_H &\leq M |\lambda|^{-\frac{1}{2}} \left\| f \right\|_H \\ \text{Hence } \left\| \rho^\alpha \frac{\partial}{\partial x_i} (p_{k_r} - \lambda I)^{-1} \right\|_H &\leq M |\lambda|^{-\frac{1}{2}}. \end{aligned}$$

From here, according to Hardy's inequality and (2.5) we conclude that

$$\left\| \rho^{2\alpha-1} (p_{k_r} - \lambda I)^{-1} \right\| \leq M_3 \sum_{i=1}^n \left\| \rho^\alpha \frac{\partial}{\partial x_i} (p_{k_r} - \lambda I)^{-1} \right\| \leq M |\lambda|^{-\frac{1}{2}}$$

Now below we introduce in H_ℓ the operator:

$$G_k(\lambda) = \sum_{r=1}^n \phi_{k_r} (p_{k_r} - \lambda I)^{-1} \phi_{k_r}, (k = v+1, \dots, \ell)$$

Here ϕ_{k_r} is the operator of multiplication by function; $\phi_{k_r}(x)$ it is easy to verify that:

$$\begin{aligned} (P_k - \lambda I) G_k(\lambda) &= I + \rho^{2\alpha-1}(x) \sum_{r=1}^n B_{k_r}(x) (P_{k_r} - \lambda I)^{-1} \phi_{k_r} \\ &+ \rho^{2\alpha}(x) \sum_{r=1}^n \sum_{i=1}^m \gamma_{k_{ir}}(x) \frac{\partial}{\partial x_i} (P_{k_r} - \lambda I)^{-1} \phi_{k_r}, \end{aligned}$$

Where $\beta_{k_{ir}}, \gamma_{k_{ir}}(x) \in L_\infty(\Omega), (x \in \Omega)$ bounded functions, $\text{supp } \beta_{k_r}, \text{supp } \gamma_{k_{ir}}(x) \subset \text{supp } \phi_{k_r}$.

Applying - then we will have:

$$\left\| (P_k - \lambda I) G_k(\lambda) - I \right\| \leq M |\lambda|^{-\frac{1}{2}} (\lambda \in \overline{\Phi}, |\lambda| \geq 1)$$

Thus for $k=v+1, \dots, \ell$ we have this representation:

$$(P_{k_r} - \lambda)^{-1} = G_k(\lambda) (I + F_k(\lambda)), \|F_k(\lambda)\| \leq M |\lambda|^{-\frac{1}{2}} (\lambda \in \overline{\Phi}, |\lambda| \geq C_0)$$

Since

$$\mu_k(x) = \overline{\mu_k(x)} > 0, (k=1, \dots, v, x \in \overline{\Omega}), \text{ from } P_k = P_k^* \geq 0, (k=1, \dots, v).$$

The above equality are well known from estimate of the resolvent of the operators P_k . Therefore in general for $k=1, \dots, \ell$ we will have

$$\left\| (P_{k_r} - \lambda)^{-1} \right\| \leq M |\lambda|^{-\frac{1}{2}}.$$

We now diagonalize the matrix function $q_{(x)}$ as follows:

$$\begin{aligned} q(x) &= U(x) \wedge (x) U^{-1}(x), \text{ Where } U(x), U^{-1}(x) \in C^2(\overline{\Omega}, \text{End } C^\ell) \\ \text{and} \\ \wedge(x) &= \text{diag} \{ \mu_1(x), \dots, U_\ell(x) \}, \end{aligned}$$

Let $\Gamma(\lambda) = U B(\lambda) U^{-1}$ that the operator $B(\lambda)$ in the direct sum.

$$H_\ell = H \oplus \dots \oplus H(\ell - \text{times}).$$

has his representation:

$$B(\lambda) = \text{diag} \{ (P_1 - \lambda I)^{-1}, \dots, (P_\ell - \lambda I)^{-1} \},$$

Such, that $\lambda \in \overline{\Phi} \setminus R_+, |\lambda| \geq C_0, (Uu)(x) = U(x)u(x), (u \in H_\ell)$.

It is easy to verify below equality:

$$(A - \lambda I) \Gamma(\lambda) = I + \rho^{2\alpha-1}(x) q_0(x) B(\lambda) U^{-10} + \rho^{2\alpha}(x) \sum_{i=1}^n q_i(x) \frac{\partial}{\partial x_i} B(\lambda) U^{-1},$$

$q_i(x) \in C(\overline{\Omega}, \text{End } C^\ell), i=0, 1, \dots, n$, Applying - we have as in

$$(A - \lambda I)^{-1} = \Gamma(\lambda) (I + F(\lambda))$$

$$\|F(\lambda)\| \leq M_\psi |\lambda|^{-\frac{1}{2}} (\lambda \in \Phi_\psi, |\lambda| \geq C_s).$$

Consequently,

$$\left\| (A - \lambda I)^{-1} \right\| \leq M_\psi |\lambda|^{-\frac{1}{2}} (\lambda \in S, |\lambda| \geq C_s).$$

The proof of Theorem 2.1 is now complete.

3. Vanishing Limit Arguments

Theorem 3.1; for sufficiently large in modulus, $\lambda \in \Phi_\psi$, where;

$$\Phi_\psi = \{ z \in C : \psi \leq |\arg z| \leq \varphi \}, \psi \in (0, \varphi), \varphi \in (0, \pi)$$

The operator A has discrete spectrum and the operator A has a finite number of (e.v.) in every set $\Phi_\psi, \psi >$

0 of the form, (2.1), therefore we will prove this relation $\lim_{j \rightarrow \infty} \arg \lambda_j = 0$.

Proof of Theorem 3.1. From the definition of the operator A and from the extension of its domain in the set^{10,12}:

$$D(A) = \left\{ u \in H_\ell \cap W_2^2, \text{loc}(\Omega)^\ell : \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (\rho^{2a} a_{ij} \frac{\partial u}{\partial x_i}) \in H_\ell \right\}$$

We know that $D(A) \subset W_2^2, \alpha(\Omega)^\ell$, and so from the compactness of the imbedding of $\{H_\ell \cap W_2^2, \text{loc}(\Omega)^\ell\} \subset H_\ell$, and since for $0 < \alpha < 1$ we know that $A : D(A) \rightarrow H_\ell$ is compact operator on the Hilbert spaces and since the compact operators on the Hilbert spaces have discrete spectrum, therefore the operator has discrete spectrum. Next for proof, $\lim_{j \rightarrow \infty} \arg \lambda_j = 0$, let the sequence of eigenvalues of the operator A in the angle $\Phi = \{z \in \mathbb{C} : |\arg z| \leq \varphi\}, \varphi \in (0, \pi)$ be denoted by $\{\lambda_i\}$ enumerated in the nondecreasing order of their absolute values and taking into account their multiplicities. Then based on the proof of the theorem 2.1 it conclude that the operator has a finite number of eigenvalues in the set $\Phi_\Psi, \Psi > 0$ let A be a given operator. Denote by $\lambda_1, \lambda_2, \dots$ the eigenvalues of A belonging to the following angle;

$$\Phi = \{z \in \mathbb{C} : |\arg z| < \varphi\} : \varphi \in (0, \pi).$$

And assume that $\lambda_j \rightarrow \infty (j \rightarrow \infty)$.

If for every closed sector $S \subset \overline{(\Phi)} \setminus \mathbb{R}_+$ with origin at zero, there exists a number $C(S) > 0$, such that for the inverse $(T - \lambda I)^{-1}$ exists and is continuous, then the following relation holds; $\lambda_j \rightarrow 0 (j \rightarrow \infty)$.

4. Asymptotic Distribution of Eigenvalues

Let denotes the number of eigenvalues of the operator located inside the angle S and does not exceeding the number .i.e., $N(t) = \text{card} \{j : |\lambda_j| \leq t\}, t > 0$, then by the assumptions of Section 1, we have the following asymptotic distribution of eigenvalues of the operator in case weak degeneracy.

Theorem 4.1. If $\alpha \in [0, 1), < \frac{1}{n}$ then we have

$$N(t) \sim (2\pi)^{-n} u_n t^{\frac{n}{2}} \int_{\Omega} \rho^{-n\alpha}(x) \mu_k^{\frac{-n}{2}}(x) (\det(a(x)))^{\frac{-1}{2}} dx,$$

Where v_n denotes the volume of the unit ball in \mathbb{R}^n and $\alpha(x) = \left(a_{ij}(x) \right)_{i,j=1}^n$.

Proof of theorem 4.1. For every natural number S,

such that

$$S \in N, S > \omega = \max \left\{ \frac{n}{2}, \frac{n-1}{2-2\alpha} \right\}, S \in (\omega, \omega+1]$$

We will have:

$$|tr(A - \lambda I)^{-S} - trUB(\lambda)U^{-1}| \leq M |\lambda|^{-\frac{1}{2}} |\Gamma(\lambda)|_S^S$$

Where the symbols $tr, 1, 1_s$ denote the trace of a trace-class operators and the σ_s -norm of the operator respectively⁴.

By using the representation of the operator B(λ) we obtain:

$$trUB(\lambda)^S U^{-1} = trB(\lambda)^S = trB_s(\lambda)$$

$$\text{Where } B_s(\lambda) = \text{diag} \{ (P_1 - \lambda I)^{-s}, \dots, (P_\ell - \lambda I)^{-s} \}.$$

If we estimate $|\Gamma(s)|_S^S$ by using and we obtain this relation:

$$\left| \sum_{i=1}^{+\infty} (\lambda_i - \lambda)^{-s} - \sum_{j=1}^{\ell} \sum_{i=1}^{+\infty} (\lambda_{ij} - \lambda)^{-s} \right| \leq M_\Psi |\lambda|^{\omega-S-\frac{1}{4}}, (\lambda \in \Phi_\Psi, |\lambda| \geq C_\Psi)$$

Where $\lambda_1, \lambda_2, \dots, \lambda_{1j}, \lambda_{2j}, \dots$ are denoted respectively the (e.v.) of the operators A and P_j . Notice that according to , there are finite number of of the using the counter integral method in the same way as in ^{4,5} for $\omega \neq 1, 2, \dots$ we can again obtain like the above relation by counter integral: i.e.

$$\sum_{i=1}^{+\infty} (\lambda_i + \tau)^{-s} = \sum_{j=1}^v \sum_{i=1}^{+\infty} (\lambda_{ij} + \tau)^{-s} + O(\tau^{\omega-s-\frac{1}{4}}), \tau \rightarrow \infty.$$

From here and from (2) keeping in mind that $\lambda_{ij} > 0$ (for $i = 1, 2, \dots, j = 1, \dots, v$). Now it is easy to establish the asymptotical formula:

$$\int_0^{+\infty} \frac{dN(t)}{(t+\tau)^s} \sim \sum_{j=1}^v \sum_{i=1}^v \int_0^{+\infty} \frac{dN_{ij}(t)}{(t+\tau)^s}, \tau \rightarrow +\infty$$

where $N_{ij}(t) = \text{card} \{j : \lambda_{ij} \leq t\}, i = 1, \dots, v$ which are well known asymptotical formulas for functions- $N_{ij}(t)$ (see for example⁶, after apply of M. V. Keldish's theorem of Tauberian-type we establish the assertion of Theorem 1,4 in applying the multi dimensional Tauberian Theorems⁷.)

5. Conclusion

In this paper we get some new results about an important differential operator on a Hilbert space. Also asymptotic distribution of eigenvalues of this kind of differential operators are proved with new methods that estimation of the resolvent of the operators is used in this paper.

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