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Comparison of Numerical Solution of 1D Hyperbolic Telegraph Equation using B-Spline and Trigonometric B-Spline by Differential Quadrature Method

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Abstract

Objectives: This paper aims to compute the approximate solution of one dimensional (1D) hyperbolic telegraph equation with appropriate primary and limiting conditions. **Methods/Statistical Analysis**: To find the approximate solution, two different modified spline basis function are used with the differential quadrature method and splines are used to compute the weighting coefficients and thus the equation is transformed to a set of first order conventional differential equations which is further solved by the SSP-RK43 method. Three test problems of this equation are simulated to establish the precision and usefulness of the proposed scheme. **Findings:** The obtained numerical results are found to be good in terms of accuracy, efficiency and simplicity. To validate the computed results using proposed scheme, various comparisons at different time levels has been done in the form of L_2 and L_∞ errors. These errors are compared and enlisted in the form of tables with computed errors enlisted in literature. **Application/Improvements:** Being an important equation of nuclear material science, one dimensional (1D) hyperbolic telegraphs equation needs to take care in the sense of better numerical solution. In this context, a successful effort has been done in this research article by proposing a hybrid numerical scheme.

Keywords: 35K57, Differential Quadrature Method, Hyperbolic Telegraphs Equation, Mathematics Subject Classification (2010): 65M06, Modified Cubic B-Spline, Modified Trigonometric B-Spline, SSP-RK43

1. Introduction

The telegraph equation is utilized to outline the reaction diffusion in numerous branches of emerging sciences. This mathematical equation premised for crucial equations of nuclear material science. It is used to demonstrate the vibrations of structures, e.g. structures, shafts, and machines. It likewise emerges in the investigation of throb blood stream in supply routes, in the 1D irregular movement of bugs along a hedge[1,2] and also play a significant part in demonstrating of numerous appropriate problems like signal investigation[3], wave propagation[4], random walk theory[5] etc. This mathematical equation is regularly utilized as a part of signal investigation for transmission and proliferation of electrical signs[6] furthermore has applications in different fields[7]. This equation is given as:

$$u_{tt}(x,t) + 2\alpha u_t(x,t) + \beta^2 u(x,t) = u_x(x,t) + f(x,t) \quad x \in [a,b] \quad t \geq 0 \tag{1}$$

With initial or starting conditions (ICs)

$$u(x,t_0) = g_1(x) \ , u_t(x,t_0) = g_2(x), \ x \in [a,b], \tag{2}$$

and with limiting conditions (BCs)

$$u(a,t) = \psi_0 \ , u(b,t) = \psi_1, \quad t > 0 \tag{3}$$

where $f, g_1, g_2, \psi_1, \psi_2$ are known functions

and $u_{tt} = \frac{\partial^2 u}{\partial t^2}, u_t = \frac{\partial u}{\partial t}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

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Eq. (1) with the coefficients $\beta = 0, \alpha > 0$ stand for a partial differential equation which is actually a damped wave equation and correspond to the telegraph mathematical equation if $\alpha > 0, \quad \beta > 0$. The telegraph equation is likewise adverted to show the combination of dissemination and wave proliferation by the properties of constrained velocity to standard warmth or mass transport condition[1].

In past few years, many numerical schemes to solve telegraph equation have been developed by many researchers. The existence of its double periodic solution was investigated in[8]. The unconditionally stable schemes for telegraph Eq. (1) was used in[9-11], etc. Further, different conditionally stable finite difference schemes were implemented to solve the telegraph equation[12-14]. In[15] finite difference for approximation of spatial derivatives and the time derivative handled by collocation scheme is used. Chebyshev cardinal functions were used for numerical simulation of the equation in[16]. Various other numerical schemes were also developed or implemented to solve the telegraph equation, Some of them are: method using interpolating scaling functions[17], Chebyshev Tau method[18], Rothe-wavelet method[1], semi-discretion methods[19], explicit difference method[14], DRBIE method[20], collocation scheme along thin plate splines radial basis function (RBFCM)[21], Polynomial Differential Quadrature Method (PDQM)[22], Cubic B-spline Quasi-interpolation (CBQ)[23], cubic B-spline collocation method (CBCM)[24], Quartic B-spline Collocation Method (QBCM)[25], and Collocation Method based on Modified Cubic B-spline (CMMCB)[26], etc.

This article is anxious with an approximate solution of 1D hyperbolic PDE with suitable starting and limiting conditions acquired by utilizing altered cubic B-spline and trigonometric B-spline with differential quadrature strategy, consequently alluded as (MCB-DQM) and (MTCB-DQM)[27]. The proposed method depends on differential quadrature technique with cubic-B-spline and trigonometric B-spline basis function. Eq. (1) is initially changed over into an arrangement of coupled Partial Differential Equations (PDEs) utilizing the transformation: $u_t = v$. The MCB-DQM and MTCB-DQM are applied to change the PDEs into a system of first order ODEs, in time, which is solved by utilizing the SSP-RK43 scheme[28]. Three test problems of this equation have been considered to exhibit the exactness and utility of the proposed technique. The maximum absolute error L_∞ and L_2 error in the MCB-DQM and MTCB-DQM approximate solution have been analogized with the error calculated by other numerical schemes accessible in the literature. The numerical results are seen to be in awesome simultaneousness with the exact solution.

The article is partitioned into various sections, in Section 2, the description of the modified cubic B-spline and modified trigonometric B-spline with Differential Quadrature Method (DQM) is given. Sections 3, discusses the procedure to utilize the proposed scheme. Three numerical test problems are given in to fabricate the significance and precision of the proposed technique in Section 4. In last section 5, summery of the paper is given in form of conclusion.

2. Description of the Method

The DQM was initially proposed by[29]. This strategy has been effectively connected to understand different one and two dimensional differential equations by utilizing various basis functions[30-33].

In DQM the derivatives of some function which are present in the given PDE are replaced with their approximate values at desired different points. Because of the dependence on weight coefficients to domain grid points, we consider N equidistance nodal points on the real axis, that is $a = x_1 < x_2 < \cdots < x_{N-1} < x_N = b$ with $x_{i+1} - x_i = h$. The solution $u(x,t)$ at knot x_i Is denoted by $u(x_i,t)$ for $i = 1, 2, 3, \ldots, N$. The approximate values of spatial derivatives are given as:

$$u_x(x_i,t) = \sum_{j=1}^{N} a_{ij} u(x_j,t), \qquad u_{xx}(x_i,t) = \sum_{j=1}^{N} b_{ij} u(x_j,t), \qquad i = 1, 2, \ldots, N \tag{4}$$

2.1 Differential Quadrature Method uses B-Spline

As B-spline have the certain nice properties like smoothness and competence to handle indigenous singularities, as B-spline basis functions are easy to implement so many researchers used cubic B-spline basis function to compute the approximate solution of physical models[26,27] definedas follows:

$$\varphi_j(x) = 1/h^3 \{ ((x - x_{j-2}))^3,$$

$$x \in [x_{j-2}, x_{j-1}) \; (x - x_{j-2})^3 - 4(x - x_{j-1})^3,$$

$$x \in [x_{j-1}, x_j) \; ((x_{j+2}) - x)^3 - 4(x_{(} \tag{5}$$

where $\{\varphi_0, \varphi_1, \dots, \varphi_N, \varphi_{N+1}\}$ generate the basis on $[a, b]$.

Lemma 1: The numerical values of φ_i and its derivatives φ_i', φ''_i at j th nodal point are evaluated as

$$\varphi_i(x_j) = \begin{cases} 4, & if \;\; i-j=0 \\ 1, & if \;\; i-j=\pm 1 \\ 0, & else \end{cases}, \quad \varphi_i'(x_j) = \begin{cases} \pm\frac{3}{h}, & i-j=\pm 1 \\ 0, & else \end{cases},$$

and

$$\varphi_i''(x_j) = \begin{cases} -\frac{12}{h^2}, & if\, i-j=0 \\ \frac{6}{h^2}, & if\, i-j=\pm 1 \\ 0, & else \end{cases}$$

The first order derivative approximation at the grid point x_i, $\quad i = 1,2,\dots,N$ is given by

$$\phi_k'(x_i) = \sum_{j=1}^{N} a_{ij}\phi_k(x_j), \qquad k = 1, 2, \dots, N. \tag{6}$$

By Lemma 1 and modified basis function mentioned in section 2.2, Eq. (6) is reduced to linear equations as

$$A\vec{a}[i] = \vec{H}[i], \text{ for } i = 1, 2, \dots, N. \tag{7}$$

and A represented as:

$$A = \begin{bmatrix} 6 & 1 & & & & & \\ 0 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 0 \\ & & & & & 1 & 6 \end{bmatrix}$$

weighting coefficient vector with respect to the points x_i, are represented as $\vec{a}[i]$, that is

$\vec{a}[i] = [a_{i1}, a_{i2}, a_{i3}, \dots a_{iN}]^T$ and the coefficient vector $\vec{h}[i] = [h_{i1}, h_{i2}, h_{i3}, \dots h_{iN}]^T$

with respect to x_i, $\quad i = 1,2,\dots,N$ are evaluated as:

$$\vec{H}[1] = \begin{bmatrix} -\frac{6}{h} \\ \frac{6}{h} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \vec{H}[2] = \begin{bmatrix} -\frac{3}{h} \\ 0 \\ \frac{3}{h} \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \vec{H}[N-1] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{3}{h} \\ 0 \\ \frac{3}{h} \end{bmatrix}, \quad \vec{H}[N] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{6}{h} \\ \frac{6}{h} \end{bmatrix}$$

2.2 Differential Quadrature Method using Trigonometric B-Spline

The cubic trigonometric B-spline basis function $\mathbf{TB_m(x)}$ for $\mathbf{m = -1, 0, 1, \dots, N+1}$ is characterized as:

where $p(x_m) = \dfrac{\sin((\mathbf{x} - \mathbf{x_m}))}{\mathbf{2}}$,

$q(x_m) = \dfrac{\sin((\mathbf{x_m} - \mathbf{x}))}{\mathbf{2}}$, and

$$w = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right) \text{ and } h = \frac{b-a}{n} \tag{8}$$

and $TB_m(x)$ is cubic trigonometric B-spline basis function with some geometric properties like

C^∞ Continuity, non-negativity and partition of unity. The values of $TB_m(x)$, $TB'_m(x)$ and $TB''_m(x)$ are given by Table 1, where:

$$a_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)}, a_2 = \frac{2}{1+2\cos(h)}, a_3 = -\frac{3}{4\sin\left(\frac{3h}{2}\right)}, a_4 = \frac{3}{4\sin\left(\frac{3h}{2}\right)}, \quad a_5 = \frac{3(1+3\cos(h))}{16\sin^2\left(\frac{h}{2}\right)\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}, \quad and$$

$$a_6 = -\frac{3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)(2+4\cos(h))}$$

Both above defined basis functions can be improved so that the resulting matrix becomes diagonally dominant. The modification of these functions can be done as follows:[27]

$B_1(x) = B_1(x) + 2B_0(x),$
$B_2(x) = B_2(x) - B_0(x),$
$B_j(x) = B_j(x) \; for \;\; j = 3, 4, \dots, N-2$
$B_N(x) = B_N(x) + 2B_{N+1}(x),$
$B_{N-1}(x) = B_{N-1}(x) - B_{N+1}(x),$

where $\{B_1, B_2, \dots, B_N\}$ form a basis on $[a, b]$

For trigonometric B-spline the system becomes $A\vec{a}[i] = \vec{R}[i]$, for $i = 1, 2, \dots, N.$

where A A represented as:

$$\begin{bmatrix} a_2+2a_1 & a_1 & 0 & 0 & . & 0 \\ 0 & a_2 & a_1 & . & . & 0 \\ 0 & a_1 & a_2 & a_1 & . & 0 \\ 0 & 0 & a_1 & a_2 & a_1 & 0 \\ 0 & . & . & . & . & 0 \\ 0 & 0 & . & . & a_1 & a_2+2a_1 \end{bmatrix} \tag{9}$$

Weighting coefficient vector with respect to the points x_i, are represented as $\vec{a}[i]$, that is

$\vec{a}[i] = [a_{i1}, a_{i2}, a_{i3}, \dots, a_{iN}]^T$ and the coefficient vector $\vec{R}[i] = [r_{i1}, r_{i2}, r_{i3}, \dots, r_{iN}]^T$

Table 1. The values of TB_m, TB'_m *and* TB''_m at different node points

$TB\backslash x$	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
TB_m	0	a_1	a_2	a_1	0
TB'_m	0	a_3	0	a_4	0
TB''_m	0	a_5	a_6	a_5	0

with respect to x_i, $i = 1,2,\dots,N$, or can be presented as:

$$\vec{R}[1] = [-2a_4, a_3 - a_4, \quad 0, \dots, 0]^T,$$

$$\vec{R}[2] = [a_4, 0, a_3, \quad 0, \dots, 0]^T,$$

$$\vec{R}[3] = [0, \quad a_4, 0, a_3, \quad 0, \dots, 0]^T, \dots, \vec{R}[N-1] = [0, \dots, \quad 0, a_4, 0, a_3]^T, \quad \vec{R}[N] = [0, \dots, 0, \quad a_4 - a_3, 2a_3]^T$$

Now we apply the well-known "Thomas Algorithm" to compute the solution of the obtained equations which provides the vector $\vec{a}[i]$, using the coefficient a_{ij}, the weighting coefficients b_{ij}, for $i = 1, 2, 3, \dots, N$, $j = 1, 2, 3, \dots, N$ are evaluated as follows:[28]

$$b_{ij} = 2a_{ij}\left(a_{ii} - \frac{1}{x_i - x_j}\right), for\ i \neq j, \quad and \quad b_{ii} = -\sum_{i=1, i\neq j}^{N} b_{ij}$$

3. Implementation of the Method

Using the transformation: $u_t = v$ the Eq. (1) is transformed to a set of PDEs as follows:

$$u_t(x,t) = v(x,t)$$
$$v_t(x,t) = u_{xx}(x,t) - 2\alpha v(x,t) - \beta^2 u(x,t) + f(x,t) \quad (10)$$

On substituting the values for the second order approximation of the space derivatives, obtained from MCB-DQM and MTCB-DQM, Eq. (10) rewritten as:

$$u_t(x_i,t) = v(x_i,t)$$
$$v_t(x_i,t) = \sum_{j=1}^{N} b_{ij} u(x_j,t) - 2\alpha v(x_i,t) - \beta^2 u(x_i,t) + f(x_i,t) \quad (11)$$

for $i = 1, 2, \dots, N$.

Hence, Eq. (11) reduces into a coupled system of first-order ODEs in time, that is,

$$\frac{du_i}{dt} = v_i \quad \text{and}$$

$$\frac{dv_i}{dt} = [\![L(u]\!]_i), \quad i = \{1,2,\dots,N\}, \quad (12)$$

where L represents the right hand side of ODEs.

The equation is solved subject to the BCs as defined in (3), and the ICs.

$$u(x_i, 0) = g_i(x_i), \quad v(x_i, 0) = g_2(x_i), \quad i \in \{1,2,\dots,N\}, \quad (13)$$

The resulting set of ODEs are solved by using SSP-RK43 scheme[28] given below:

$$u^{(1)} = u^m + \frac{\Delta t}{2} L(u^m)$$

$$u^{(2)} = u^{(1)} + \frac{\Delta t}{2} L\left(u^{(1)}\right)$$

$$u^{(3)} = \frac{2}{3}u^m + \frac{u^2}{3} + \frac{\Delta t}{6} L\left(u^{(2)}\right)$$

$$u^{m+1} = u^{(3)} + \frac{\Delta t}{2} L\left(u^{(3)}\right)$$

and consequently the solutions $u(x,t)$, at the required time level are obtained.

4. Numerical Experiments

For numerical discussion, three test problems are considered to obtain the approximate solutions by MCB-DQM and MTCB-DQM. The L_2 and L_∞ error norms are calculated using the exact solution and formulas to calculate these errors are defined in Eq. (14).

$$L_2 = \left(h \sum_{j=1}^{N} \left[u_j^{exact} - u_j^*\right]^2\right)^{\frac{1}{2}} \quad \text{and}$$

$$L_\infty = \max_{j=1}^{N} \left|u_j^{exact} - u_j^*\right|, \quad (14)$$

where u_j^* represent the numerical solution at node j.

Example 1: The telegraph equation over the region $[0, \pi]$ is considered with following conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \geq 0$$

and the function

$$f(x,t) = (2 - 2\alpha + \beta^2)e^{-t}\sin(x)$$

The exact solution[19,20] is given by:

$$u(x,t) = e^{-t}\sin(x) \quad (15)$$

The comparison of L_2 and L_∞ errors at different time levels $t \leq 2$ is done for $\alpha = 2, \quad \beta = \sqrt{2}$, $\alpha = 3, \quad \beta = \sqrt{2}$

(a) For $\alpha = 2,\quad \beta = \sqrt{2}$ the L_2 and L_∞ errors with $h = 0.02$, $\Delta t = 0.01$ are compared with the errors due to the well-known earlier schemes: CMMCB[26] and RBFCM[21], and are reported in Table 2. It is evident that the errors are decreasing with increment in time, and the numerical results are more accurate than numerical solutions obtained by CMMCB[26] and RBFCM[21]. The physical conduct of solutions obtained by MCB-DQM and MTCB-DQM at various time levels $t \le 2$ is depicted in Figure 1.

Table 2. Comparison of L_2 and L_∞ errors at $t \le 2$ with $h = 0.02$ and $\Delta t = 0.01$.

t	L_2			
	$MTCB-DQM$	$MCB-DQM$	$CMMCB^{26}$	$RBFCM^{21}$
0.5	$1.4022E-07$	$2.1570E-07$	$2.3328E-06$	$7.9491E-05$
1.0	$1.059E-07$	$1.6491E-07$	$4.3667E-06$	$1.4554E-04$
1.5	$7.2312E-08$	$1.1434E-07$	$4.7817E-06$	$1.5895E-04$
2.0	$5.0662E-08$	$8.0521E-08$	$4.2706E-06$	$1.4185E-04$
	L_∞			
0.5	$1.9891E-07$	$3.5160E-07$	$1.8612E-06$	$8.3721E-06$
1.0	$1.1694E-07$	$2.1430E-07$	$3.4839E-06$	$1.5680E-05$
1.5	$7.1148E-08$	$1.3045E-08$	$3.8251E-06$	$1.7412E-05$
2.0	$4.3259E-08$	$7.9350E-08$	$3.4073E-06$	$1.5813E-05$

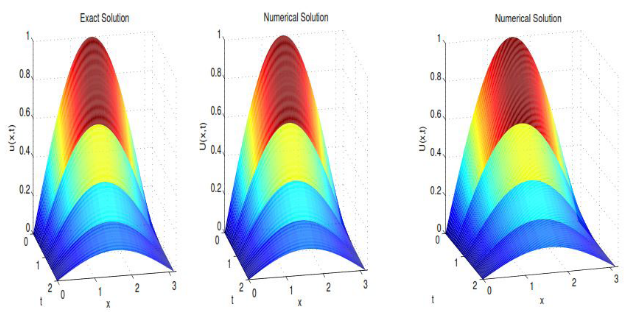

Figure 1. At different time levels $t \le 2$ the performances of exact (Left) and approximate solution of Example 1 with $\alpha = 2,\quad \beta = \sqrt{2}$ using trigonometric B-spline (Middle) and B-spline (Right).

(b) For $\alpha = 3,\quad \beta = \sqrt{2}$ the L_2 and L_∞ errors with $h = 0.02$, $\Delta t = 0.0001$ are compared with the errors by RBFCM[21], and are reported in Table 3. It is evident that the errors are decreasing as time increases (also, by decreasing the values of h), and computed results are more accurate than the results obtained in[21]. The physical behaviour of the MCB-DQM and MTCB-DQM solutions for $\Delta t = 0.0001$ are depicted in Figure 2 at various time levels $t \le 2$.

Table 3. Comparison of L_2 and L_∞ errors at $t \le 2$ with $h = 0.02$ and $\Delta t = 0.0001$.

t	L_2		
	$MTCB-DQM$	$MCB-DQM$	$RBFCM^{21}$
0.5	$5.5700E-06$	$6.22E-05$	$1.55E-05$
1.0	$9.7571E-06$	$1.05E-05$	$2.71E-05$
1.5	$1.0903E-05$	$1.16E-05$	$3.03E-05$
2.0	$1.0407E-05$	$1.11E-05$	$2.92E-05$
	L_∞		
0.5	$4.4591E-06$	$4.46E-06$	$1.41E-04$
1.0	$7.8127E-06$	$7.81E-06$	$2.46E-04$
1.5	$8.7306E-06$	$8.74E-06$	$2.74E-04$
2.0	$8.3216E-06$	$8.53E-06$	$2.62E-04$

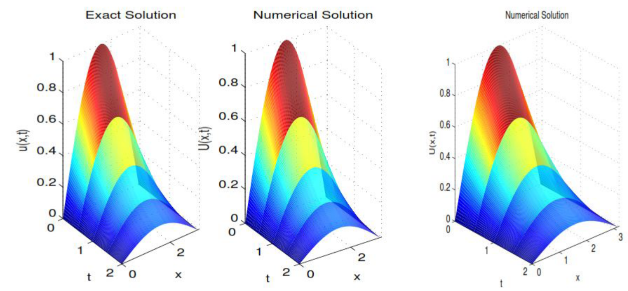

Figure 2. At different time levels $t \le 2$ the performances of exact (Left) and approximate solution of Example 2 with $\alpha = 3,\quad \beta = \sqrt{2}$ using trigonometric B-spline (Middle) and B-spline (Right).

Example 2: The telegraph equation is considered over the region $[0, 1]$ for $\alpha = 0.5,\quad \beta = 1$ with conditions.

$$u(x, 0) = 0, \qquad u_t(x, 0) = 0,$$

$$u(0, t) = 0, \qquad u(1, t) = 0, \qquad t \ge 0$$

and the function

$$f(x, t) = (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2 e^{-t}$$

The analytic solution of the equation[18,21] is given as:

$$u(x, t) = (x - x^2)t^2 e^{-t}$$

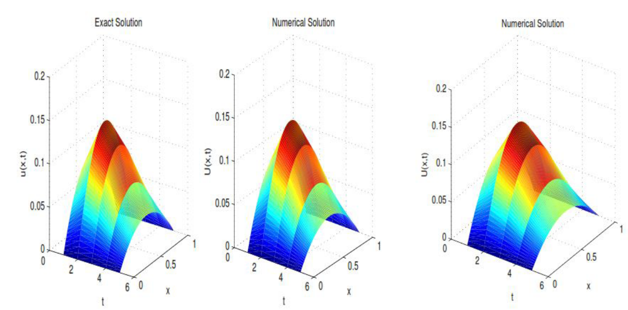

Figure 3. At different time levels $t \le 5$ the performances of exact (Left) and approximate solution of Example 2 with $\Delta t = 0.0001,\quad h = 0.0125$ using trigonometric B-spline (Middle) and B-spline (Right).

The comparison of the L_2 and L_∞ errors is done at different time levels with $\Delta t = 0.0001$ and $h = 0.0125$ with the errors obtained by the earlier schemes: CMMCB[26], RBFCM[21] and QBCM[25], and are presented in Table 4 and it can be seen that the obtained results are better than the results given in[21,25,26]. The physical behaviour of the MCB-DQM and MTCB-DQM solutions at various time levels $t \leq 5$ are shown in Figure 4.

Table 4. Comparison of L_2 and L_∞ errors at different time levels ≤ 5 with the errors due to well-known earlier schemes

t	$MTCB-DQM$ $h = 0.0125$, $\Delta t = 0.001$		$MCB-DQM$ $h = 0.0125$, $\Delta t = 0.001$		$QBCM^{25}$ $h = 0.005, \Delta t = 0.01$
	L_2	L_∞	L_2	L_∞	L_∞
1	$4.00E-05$	$5.40E-05$	$4.20E-05$	$5.58E-05$	$1.9175E-04$
2	$2.51E-06$	$5.32E-06$	$6.49E-06$	$9.71E-06$	$1.1387E-04$
3	$5.72E-06$	$8.48E-06$	$3.96E-06$	$7.00E-06$	$1.7053E-04$
4	$1.40E-05$	$2.00E-05$	$1.19E-05$	$1.76E-05$	$2.0271E-04$
5	$5.26E-06$	$7.49E-06$	$4.23E-06$	$3.01E-06$	$9.8405E-05$

t	$CMMCB^{26}$ $h = 0.01$, $\Delta t = 0.001$		$RBFCM^{21}$ $h = 0.01$, $\Delta t = 0.001$	
	L_2	L_∞	L_2	L_∞
1	$4.55E-05$	$1.43E-05$	$1.43E-04$	$1.84E-05$
2	$1.43E-05$	$8.08E-05$	$8.08E-05$	$1.07E-05$
3	$6.42E-06$	$1.29E-05$	$1.29E-04$	$1.81E-05$
4	$8.92E-06$	$1.18E-05$	$1.18E-04$	$1.64E-05$
5	$3.01E-06$	$7.55E-06$	$7.55E-05$	$1.04E-05$

Example 3: The telegraph equation over the region [0, 2] is considered with conditions.

$$u(0,t) = \tan\left(\frac{t}{2}\right), \qquad u(2,t) = \tan\left(\frac{2+t}{2}\right), \qquad t \geq 0$$

and the function

$$f(x,t) = \alpha\left(1 + \tan^2\left(\frac{x+t}{2}\right)\right) + \beta^2 \tan^2\left(\frac{x+t}{2}\right)$$

The analytical solution[15]

$$u(x,t) = \left(\frac{x+t}{2}\right)$$

Numerical solutions are computed for $\alpha = 10, \beta = 5$ taking $\Delta t = 0.001, 0.0001$ and $h = 0.025$ at diverse time levels $t \leq 1$. The comparison of L_∞ error is done at different time levels with earlier schemes[23,25,26] and are reported in Table 5. It is evident from Table 5 and 6 that obtained results are matched with the analytical solution and also better than the previously obtained results. The behaviour of the MCB-DQM and MTCB-DQM numerical solutions are depicted physically in Figure 4 at different time level $t \leq 1.0$, taking $\Delta t = 0.0001$ and $h = 0.025$ $h = 0.025$.

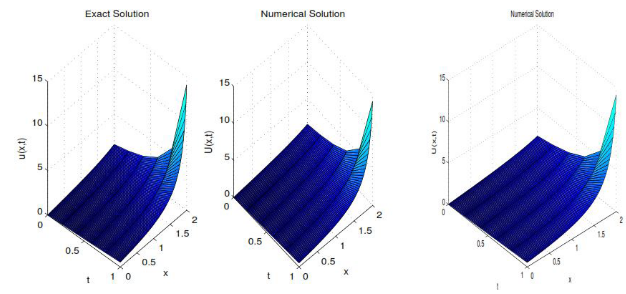

Figure 4. At different time levels $t \leq 1$ the performances of exact (Left) and approximate solution of Example 3 with $\Delta t = 0.0001$ and $h = 0.025$ using trigonometric B-spline (Middle) and B-spline (Right).

5. Conclusion

Solution of the hyperbolic telegraph equation is obtained numerically in this paper using two different schemes named as, MCB-DQM and MTCB-DQM. These schemes are based on the DQM combined with modified cubic B-spline and modified cubic trigonometric B-spline as basic functions. On implementing the schemes and substituting the derivatives, set of ODEs are attained, which is solved using SSP RK43. The efficiency and precision of the proposed method is revealed by three test problems. The numerical result, L_2 and L_∞ errors are compared with numerical solutions from literature and are found to be in decent agreement with formerly obtained results. The advantage of the developed methods is there ease to implement and reduced data complexity as compared to the present schemes.

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Table 5. Comparison of L_∞ errors in Example 3 at different time levels $t \leq 1$ with the errors in the earlier schemes

Schemes	$(h, \Delta t)$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
$MCB-DQM$	(0.025, 0.0001)	4.85E-05	4.91E-05	2.26E-05	4.33E-04	1.12E-02
$MTCB-DQM$	(0.025, 0.0001)	2.42E-04	3.80E-04	6.98E-04	1.73E-03	1.64E-02
$CMMCB$[26]	(0.020, 0.0001)	3.47E-05	5.34E-05	9.47E-05	1.87E-04	5.87E-04
$MCB-DQM$	(0.025, 0.001)	9.62E-04	1.47E-03	2.56E-03	5.61E-03	1.84E-02
$MTCB-DQM$	(0.025, 0.001)	2.42E-03	3.80E-03	6.97E-03	1.72E-02	9.92E-02
$CMMCB$[26]	(0.020, 0.001)	2.63E-04	6.99E-04	1.48E-03	3.40E-03	1.11E-02
CBQ[23]	(0.005, 0.001)	1.89E-04	3.99E-04	7.97E-04	1.87E-03	8.01E-03
$QBCM$[25]	(0.001, 0.001)	2.77E-04	7.07E-04	1.38E-03	3.09E-03	1.34E-02

Table 6. Comparison of L_2 errors in Example 3 at different time levels $t \leq 1$ with the errors in the earlier scheme[26]

Schemes	$(h, \Delta t)$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
$MCB-DQM$	(0.025, 0.0001)	1.31E-05	1.48E-05	1.01E-05	8.56E-05	1.93E-03
$MTCB-DQM$	(0.025, 0.0001)	4.85E-05	8.18E-05	1.40E-04	3.30E-04	3.33E-03
$CMMCB$[26]	(0.020, 0.0001)	5.03E-05	9.51E-05	2.20E-04	7.82E-04	7.92E-03
$MCB-DQM$	(0.025, 0.001)	2.41E-04	3.82E-04	6.45E-04	1.31E-03	3.97E-03
$MTCB-DQM$	(0.025, 0.001)	5.82E-04	1.02E-03	1.82E-03	3.95E-03	1.76E-02
$CMMCB$[26]	(0.020, 0.001)	1.87E-04	4.88E-04	9.48E-04	1.87E-03	5.09E-03

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