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Queue Length and Busy Period Analysis for the M/G/1 Queue with Negative Arrivals

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Abstract

In this paper, the queue lengths and the busy period lengths of the M/G/1 queueing systems with negative arrivals are analyzed. Two types of negative arrivals are considered. One type is negative customers and the other type is disasters. When a negative customer arrives to a system, one positive customer is removed if the number of positive customers is more than one. In particular, we assume the RCH (Removal of a Customer at the Head) type of negative customers which represent a kind of work-canceling signal to the positive customer in service. On the other hand, disasters get rid of all customers in the system. In this paper, the Probability Generating Function (PGF) of the stationary queue length and busy period length of M/G/1 queue with both negative customers and disasters are derived.

Keywords: Busy Period, M/G/1 Queue, Negative Arrivals, Queue Length

1. Introduction

In classical single server queues, all customers who arrive the system are intended to be getting served. Suppose, however, that customers can act as work-canceling signals or reset orders that clear the queue at once. This is the notion of negative arrivals introduced by Gelenbe[6], which is inspired by the neural networks including an inhibition signal. Queueing systems with negative arrivals can provide a better account of the operations of many systems, since they consider possible failure events in a process such as removing scrapped material and server breakdown leading to loss of every work in a queue.

Negative arrivals can in general be divided into two types. One is a negative customer, which puts one ordinary customer, also called a positive customer out when it arrives at the system. The other is a disaster, which causes all customers in the system to be removed immediately. If the system is empty, no customers are removed by negative customers or disasters. In particular, negative customers fall into two types depending on which customers in the system are removed: 'the Removal of Customer at the Head (RCH)' and 'the Removal of Customer at the End (RCE)' (see Harrison and Pitel[8]).

Queues with negative arrivals have been widely investigated over past few years (see[2-3,5,7,9-16,18-25] and references therein). Negative arrivals in a Wireless Sensor Network (WSN) are applicable to the network with unreliable connections where packets can be lost by external causes, attacks or shocks. A WSN can be used for military purpose to gather intelligence in battlefield. For example, a WSN can be applied to detect and track enemy troop movements nearby. Jamming signals may be send by an enemy to jam wireless communications. We call this kind of attack as a jamming attack. The radio frequencies which sensor nodes in the WSN are in use are interfered with the jamming signals. The data packets which are being transmitted through the frequency can be discarded and may need to be retransmitted. Moreover, sudden environmental changes such as the magnetic field disturbance due to solar flares and the attenuation caused by heavy snow or hard rain can make wireless connections unreliable.

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Extensive reviews regarding queue with negative arrivals can be also found in Artalejo[1] and Do[4].

Most of the previous studies on queueing systems with negative arrivals have considered models with negative customers and disasters separately. The solving process of the queueing model under the assumption of both negative customers and disasters such as derivation of closed form of performance measures would become more difficult. In many practical systems, however, we may encounter the situations where either the customer is being served or all customers in the system are removed by outside interference. For this reason, we take models with both negative customers and disasters into consideration simultaneously. To derive a closed form of stationary queue length distribution, we combine both a supplementary variable technique and a busy period analysis.

The remainder of the paper is organized as follows: In section 2, the queueing model is presented with notations. In section 3, the length of busy period is analyzed. The steady-state distributions for the number of customers in the system with negative arrivals are provided in section 4. The conclusion of this research is given in the final section.

2. Model Description

This paper assumes that all events can only occur at exact slot boundaries. Exponential random variables $\{A_n, n \geq 1\}$ denote the interarrival times of positive customers are independent and identically distributed (iid). The rates of the exponential variables are assumed to be λ. Also, interarrival times of negative customers $\{C_n, n \geq 1\}$ and disasters $\{D_n, n \geq 1\}$ are iid random variables with an exponential distribution of rate η and δ, respectively. Service discipline is a First-In, First-Out (FIFO) and service times $\{S_n, n \geq 1\}$ are assumed to follow general distribution. Customers are served on a First-Come, First-Served (FCFS) basis, and service times $\{S_n, n \geq 1\}$ are iid random variables with general distribution. Note that $\{A_n, n \geq 1\}$,$\{C_n, n \geq 1\}$, $\{D_n, n \geq 1\}$, and $\{S_n, n \geq 1\}$ are mutually independent. Furthermore, it is assumed that when a negative customer arrives to the system, the customer in service is removed. That is, RCH discipline is adopted.

The supplementary variable technique is used to derive the stationary queue length Probability Generating Function (PGF), $Q^*(z)$. The following notations and probabilities will be used commonly in this section:

$s(x)$, $S^*(\theta)$: Probability Density Function (PDF) and Laplace Stieltjes-transform (LST) of service times,

$N(t)$: The number of customers at time t ,

$S_R(t)$: The remaining service time at time t ,

$$P_0(t) = P\{N(t)=0\},$$
$$P_n(t,x)dt = P\{N(t)=n, x<S_R(t)<x+dx\}, \quad n \geq 1,$$
$$P_n(x) = \lim_{t\to\infty} P_n(t,x),$$

$$P_n = \int_0^\infty P_n(x)dx,$$
$$P_n^*(\theta) = \int_0^\infty e^{-\theta x} P_n(x)dx,$$
$$P(z,0) = \sum_{n=1}^\infty P_n(0)z^n, \qquad |z| \leq 1,$$
$$P^*(z,\theta) = \sum_{n=1}^\infty P_n^*(\theta)z^n, \qquad |z| \leq 1,$$

$$Q^*(z) = \sum_{n=0}^\infty P_n z^n = P_0 + P^*(z) = P_0 + P^*(z,\theta)\big|_{\theta=0}$$

3. Busy Period Analysis

In the system under study, customers leave a system in three cases:

1. Service completion (S),
2. Arrival of a negative customer (C),
3. Occurrence of a disaster (D). In cases 1 and 2, only one customer can leave the system. In case 3, however, every customer leaves the system at once. A busy period of the M/G/1 queue with negative arrivals can be terminated by two causes. The first is a type-I busy period in which every customer leaves the system one by one by either 1 or 2 until the system becomes empty. The other is a type-II busy period in which a disaster terminates a busy period. Before dealing with a type-I case, the modified service time is defined as the actual amount of service that a positive customer receives before departing the system either by 1 or 2. Let S_m and $S_m^*(\theta)$ denote the modified service time and its LST, respectively, then we get

$$\begin{aligned} S_m^*(\theta) &= P(S<C)\cdot E\left[e^{-\theta S} \mid S<C\right] + P(S>C)\cdot E\left[e^{-\theta C} \mid S>C\right] \\ &= S^*(\eta)\cdot\frac{S^*(\theta+\eta)}{S^*(\eta)} + \left\{1-S^*(\eta)\right\}\cdot\left(\frac{\eta}{\theta+\eta}\right)\left\{\frac{1-S^*(\theta+\eta)}{1-S^*(\eta)}\right\} \\ &= \frac{\eta+\theta\cdot S^*(\theta+\eta)}{\theta+\eta}. \end{aligned} \tag{1}$$

The LST of the type-I busy period, denoted by $T^*(\theta)$, can be obtained by using (1). The LST of a busy period length in a standard M/G/1 queue, $B^*(\theta)$, is known as (see Takagi[17])

$$B^*(\theta) = S^*\left(\theta + \lambda - \lambda B^*(\theta)\right) \tag{2}$$

If we replace $S^*(\theta)$ in (2) with $S_m^*(\theta)$ of (1), then we can express the LST of a type-I busy period as follows:

$$T^*(\theta) = S_m^*\left(\theta + \lambda - \lambda T^*(\theta)\right).$$

A type-II busy period is then an interarrival time of a disaster. Finally, we can represent the LST of a busy period length of our model(B_m)as the minimum distribution of T (type-I) and D (type-II) as follows:

$$\begin{aligned} B_m^*(\theta) &= P(T<D)\cdot E\left[e^{-\theta T} \mid T<D\right] + P(T>D)\cdot E\left[e^{-\theta D} \mid T>D\right] \\ &= T^*(\delta)\cdot\frac{T^*(\theta+\delta)}{T^*(\delta)} + \left\{1-T^*(\delta)\right\}\cdot\left(\frac{\delta}{\theta+\delta}\right)\left\{\frac{1-T^*(\theta+\delta)}{1-T^*(\delta)}\right\} \\ &= \frac{\delta+\theta\cdot T^*(\theta+\delta)}{\theta+\delta}. \end{aligned} \tag{3}$$

Remark 1: We can get an expected length of the busy period of our model by differentiating (3) as follows:

$$E[B_m] = -\frac{d}{dz}B_m^*(\theta)\big|_{z=0} = \frac{1-T^*(\delta)}{\delta}.$$

Since the expected duration of the idle period is λ^1, P_0 can be obtained by the renewal reward theorem as follows:

$$P_0 = \frac{\lambda^{-1}}{\lambda^{-1}+\delta^{-1}\left\{1-T^*(\delta)\right\}} = \frac{\delta}{\delta+\lambda\left\{1-T^*(\delta)\right\}}. \tag{4}$$

Remark 2: Since P_0 in (4) is positive, it is true that $\delta>0$. This condition is necessary for the system to be stable. Also, if $\delta> 0$ then the system under study is stable. This is a sufficient condition. Hence, the system is stable if and only if $\delta> 0$.

4. Queue Length Analysis

In Now, the supplementary variable technique is used to derive the stationary PGF of the queue length at an arbitrary time. The steady state system equations are

$$\begin{aligned} P_0(t+dt) &= P_0(t)(1-\lambda\cdot dt) + P_1(t,0)dt \\ &\quad + P_1\eta\cdot dt + \left\{1-P_0(t)\right\}\delta\cdot dt, \end{aligned} \tag{5}$$

$$\begin{aligned} P_0(t+dt,x-dt)dt &= P_1(t,x)\left\{1-\left(\lambda+\eta+\delta\right)dt\right\} \\ &\quad + P_0(t)\lambda\cdot dt\cdot s(x) \\ &\quad + P_2(t,0)\cdot s(x)dt + P_2\eta\cdot dt\cdot s(x), \end{aligned} \tag{6}$$

$$\begin{aligned} P_n(t+dt,x-dt)dt &= P_n(t,x)\left\{1-\left(\lambda+\eta+\delta\right)dt\right\} \\ &\quad + P_{n-1}(t,x)\cdot\lambda\cdot dt \\ &\quad + P_{n+1}(t,0)\cdot s(x)dt \\ &\quad + P_{n+1}\eta\cdot dt\cdot s(x), \qquad n\ge 2. \end{aligned} \tag{7}$$

First, we can get the following differential equations using (5) through (7):

$$\begin{aligned} \frac{d}{dt}P_0(t) &= -\lambda P_0(t)dt + P_1(t,0) + \eta P_1 + \left\{1-P_0(t)\right\}\delta, \\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_1(t,x) &= -\left(\lambda+\eta+\delta\right)P_1(t,x) + \lambda P_0(t)s(x) \\ &\quad + P_2(t,0)s(x) + \eta P_2 s(x), \\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_n(t,x) &= -\left(\lambda+\eta+\delta\right)P_n(t,x) + \lambda P_{n-1}(t,x) \\ &\quad + P_{n+1}(t,0)s(x) + \eta P_{n+1}s(x), \qquad n\ge 2. \end{aligned} \tag{8}$$

The next step is to obtain the steady state relations by taking a limit to (8) as follows:

$$\left(\lambda+\delta\right)P_0 = P_1(0) + \eta P_1 + \delta, \tag{9}$$

$$\begin{aligned} -\frac{d}{dx}P_1(x) &= -\left(\lambda+\eta+\delta\right)P_1(x) + \lambda P_0 s(x) \\ &\quad + P_2(0)s(x) + \eta P_2 s(x), \\ -\frac{d}{dx}P_n(x) &= -\left(\lambda+\eta+\delta\right)P_n(x) + \lambda P_{n-1}(x) \\ &\quad + P_{n+1}(0)s(x) + \eta P_{n+1}s(x), \quad n\ge 2. \end{aligned} \tag{10}$$

After then, we take LST of (10), and get

$$\begin{aligned} -\theta P_1^*(\theta) + P_1(0) &= -\left(\lambda+\eta+\delta\right)P_1^*(\theta) \\ &\quad + S^*(\theta)\left\{\lambda P_0 + P_2(0) + \eta P_2\right\}, \\ -\theta P_n^*(\theta) + P_n(0) &= -\left(\lambda+\eta+\delta\right)P_n^*(\theta) + \lambda P_{n-1}^*(\theta) \\ &\quad + S^*(\theta)\left\{P_{n+1}(0) + \eta P_{n+1}\right\}, \quad n\ge 2. \end{aligned} \tag{11}$$

After multiplying (11) by z^k and summing over k = 1,2,…, we get (9) after simplification of the result.

$$\begin{aligned} -\theta P^*(z,\theta) + P(z,0) &= -\left(\lambda+\eta+\delta\right)P^*(z,\theta) \\ &\quad + \lambda z P_0 S^*(\theta) + \lambda z P^*(z,\theta) \\ &\quad + \left\{z^{-1}P(z,0) - P_1(0)\right\}S^*(\theta) \\ &\quad + \left\{z^{-1}P^*(z) - P_1\right\}\eta S^*(\theta), \end{aligned}$$

$$P^*(z,\theta)(\theta_0-\theta)$$
$$= z^{-1}P(z,0)\{S^*(\theta)-z\}$$
$$-S^*(\theta)\{\lambda P_0(1-z)-\delta(1-P_0)-z^{-1}\eta P^*(z)\}, \quad (12)$$

where $\theta_0 = (\lambda+\eta+\delta)-\lambda z$.

To obtain $P(z,0)$, we substitute $\theta = \theta_0$ in (12). Thus,

$$P(z,0) = \frac{zS^*(\theta_0)\{\lambda P_0(1-z)-\delta(1-P_0)-z^{-1}\eta P^*(z)\}}{\{S^*(\theta_0)-z\}}. \quad (13)$$

Substituting (13) into (12) then yields

$$P^*(z,\theta)(\theta_0-\theta)\{S^*(\theta_0)-z\}$$
$$= \{S^*(\theta)-S^*(\theta_0)\}\{\lambda zP_0(1-z)-\delta z(1-P_0)-\eta P^*(z)\}. \quad (14)$$

Finally, we obtain $P^*(z)$ by substituting $\theta = 0$ into (14). Note that $S^*(0) = 1$. We finally get

$$P^*(z) = \frac{\{\lambda zP_0(1-z)-\delta z(1-P_0)\}\{1-S^*(\theta_0)\}}{\theta_0\{S^*(\theta_0)-z\}+\eta\{1-S^*(\theta_0)\}}. \quad (15)$$

Noted that P_0 in (15) is the only unknown that cannot be determined by $P^*(1) = 1 - P_0$. However, by analyzing a regeneration cycle, we can obtain P_0 in (4). The stationary PGF of the queue length at an arbitrary epoch, $Q^*(z)$, can be given by

$$Q^*(z) = P_0 + \frac{\{\lambda zP_0(1-z)-\delta z(1-P_0)\}\{1-S^*(\theta_0)\}}{\theta_0\{S^*(\theta_0)-z\}+\eta\{1-S^*(\theta_0)\}},$$

where

$$P_0 = \frac{\delta}{\delta+\lambda\{1-T^*(\delta)\}}.$$

5. Conclusion

In this paper, we analyzed queueing systems with negative arrivals. Negative arrivals include negative customers which act as inhibition signals in neural networks and disasters. Disasters represent to damaging events that remove all the workload or data packets in a system by using a virus or resetting order. We obtained the PGF of the length of a busy period by regeneration cycle analysis. Also, using the supplementary variable technique, we derived the PGF of the queue length distribution.

Our research addresses the queueing models with both negative customers and disasters simultaneously, that is differentiated from other previous works. Our model and results can be useful to estimate failure costs occurred by sudden shocks to a machine or scraps in various industrial sites.

In future work, we may consider the discrete time queue with negative arrivals. Also, more general type of negative arrival can be addressed besides negative customers and disasters. An additional type of negative arrival is the case of random size removal. When negative arrivals happen, random numbers of customers are deleted from the queueing systems. Thus, it is feasible for future work is to analyze the discrete-time queue with the random size removal of negative arrival.

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