Strongly Nonlinear Free Vibration Analysis of Beams using Modified Homotopy Perturbation Method subjected to the Nonlinear Thermal Loads

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Abstract

Objectives: In this study, large amplitude e free vibration behavior of Euler-Bernoulli beam subjected to the nonlinear thermal loads and resting on a Pasternak foundation is investigated. **Methods:** The Hamilton's principle is used to derive the beam governing partial differential equation of motion. By implementing the Galerkin's method and applying the clamped-clamped boundary condition, the partial differential equation is converted to an ordinary nonlinear differential equation. **Results:** Because of the large coefficient of the nonlinear term, the Modified Homotopy Perturbation Method (MHPM) is used to solve the obtained equation. The effect of nonlinear thermal load on the system nonlinear vibration behavior is studied. **Applications:** The results show that although increasing the nonlinear thermal load coefficients decreases both linear and nonlinear frequency, but it increases the frequency ratio.

Keywords: Euler-Bernoulli Beam, Modified Homotopy Perturbation Method, Nonlinear Thermal Load, Pasternak Foundation, Strongly Nonlinear Vibration

1. Introduction

Most of the physical phenomena and engineering problems occur in nature in the forms of nonlinear differential systems. Many structures such as high-rise buildings, long span bridges and aerospace vehicles can be modeled as a beam and by increasing the amplitude of oscillations, the governed equation of motion can be obtained as a nonlinear ODE. The common techniques for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. Some well-known perturbation methods are the Krylov Bogoliubov Mitropolskii (KBM)[1–5] method, the Lindstedt-Poincare (LP) method[6–8] and the method of multiple time scales[9]. All of these classical perturbation methods are based on assuming a small parameter which exists in the equation. In[10] has investigated the homotopy perturbation technique. In another paper, in[11] has developed a coupling method of a homotopy perturbation technique and a perturbation technique for strongly nonlinear problems. Recently, in[12] has also presented a new interpretation of homotopy perturbation method for strongly nonlinear differential systems. In[13] proposed a new perturbation technique to solve the nonlinear un-damped Duffing equation in which the maximum relative error at the first order approximation is less than 7%. In[14] presented a method called MHPM which can solve strongly nonlinear problems more accurately. They show that the maximum relative error at the first order and second order of MHPM is less than 2.22% and 0.03%, respectively. Newly, in[15] studied nonlinear free vibration of laminated composite thin beams on nonlinear elastic foundation with elastically restrained against rotation edges by Differential Quadrature (DQ) approach. They developed a finite element program to verify the results of the DQ approach and also they studied the effects of different parameters on the ratio of nonlinear to linear natural frequency. In[16] investigated the large amplitude

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free vibration of a doubly clamped micro beam. They used Hamilton's principle for deriving the partial differential equation of motion. Then, they used the method of multiple scales to determine a second-order perturbation solution for their obtained nonlinear ODE. As described classical perturbation techniques like as multiple scales method strongly rely on the assumption of the small parameter. However, as mentioned, the coefficient of the nonlinear term in the governing equation of the beam motion is so large therefore the classical perturbation techniques wouldn't lead to a valid solution. In this paper the large amplitude free vibration behavior of the beam with the clamped-clamped ends, resting on a Pasternak foundation and subjected to nonlinear thermal load, is investigated. Moreover, the effect of Pasternak foundation is considered in calculating the Lagrangian. To this end, first the Hamilton's principle is used to derive the partial differential equation of the beam response. Then implementing the Galerkin's method under the mentioned boundary condition, the partial differential equation is converted to an ordinary nonlinear differential equation. Since, the nonlinear coefficient is large, so the MHPM is carried out to solve the obtained nonlinear ODE. It is assumed that only the fundamental mode is excited. Comparison between second order approximation of the MHPM and available results in the literature show that a second order approximation of the MHPM leads to a highly accurate solution that is valid for a wide range of vibration amplitudes.

2. Equation of Motion

A schematic of Euler-Bernoulli beam with a length of L, cross-sectional area of A, density of ρ, area moment of inertia of I and the elasticity modulus of E, resting on a Pasternak foundation, is shown in Figure 1. The strain-displacement relations for a beam undergoing large deflections are as[17]:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \kappa_x = -\frac{\partial^2 w}{\partial x^2} \tag{1}$$

Where u is the longitudinal displacement, w is the lateral displacement, and x is the longitudinal coordinate. Neglecting the axial inertia, the strain energy (U) and the kinetic energy (T) of the beam is given by:

$$U = \frac{1}{2}\int_0^L \left(EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right)dx \tag{2}$$

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 dx \tag{3}$$

Where k_1 and k_2 are linear and nonlinear foundation stiffness, respectively.

Employing the Hamilton's principle, the governing equations including the effects of mid-plane stretching for Euler-Bernoulli beam is given by:

$$\frac{\partial}{\partial x}\left(EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right) = 0 \tag{4}$$

$$EI\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}\left(EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial w}{\partial x}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = -k_L w - k_{NL}w^3 \tag{5}$$

By integrating Equation (4) and substituting the result into Equation (5), one obtains the equation of motion as:

$$EI\frac{\partial^4 w}{\partial x^4} + F_{Th}\frac{\partial^2 w}{\partial x^2} - \frac{EA}{2L}\left(\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2} + k_L w + k_{NL}w^3 + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{6}$$

Where F_{Th} the nonlinear thermal force is due to the nonlinear thermal stress. The conductivity of any material is reciprocal of its resistance and is denoted as $\alpha = 1/D$ where D is the resistivity of the material. The relation between the temperature rising value, ΔT, and the resistance is considered as:

$$D = D_0 + D_0\alpha\Delta T \tag{7}$$

Where D_0 is the resistance at room temperature. Thus, the temperature and the resistivity of material are dependent. The nonlinear thermal force is defined as[18]:

$$F_{Th} = EA\alpha\Delta T + hA\alpha^2\Delta T^2 \tag{8}$$

where:

$$h = h_1(1 - 2v) - 2h_2(v^2 - 1) + h_3 v^2 \tag{9}$$

In the above equation, h_1, h_2, h_3 are Murnaghan's constants and v is Poisson's ratio.

The dimensionless quantities are defined as:

$$\hat{t} = \omega t, \qquad \hat{x} = \frac{x}{L}, \qquad \hat{w} = \frac{w}{L}, \qquad \omega = \beta^2\sqrt{\frac{EI}{\rho A}} \tag{10}$$

Where ω is the linear natural frequency of the beam with clamped-clamped boundary condition. Substitution of Equation (10) into Equation (6) and using the chain rule for differentiation yields:

$$\frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + \frac{F_{Th}}{EA}\left(\frac{L}{r}\right)^2 \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} - \frac{1}{2}\left(\frac{L}{r}\right)^2 \left(\int_0^1 \left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 d\hat{x}\right)\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} +$$
$$\frac{k_L L^2}{EA}\left(\frac{L}{r}\right)^2 \hat{w} + \frac{k_{NL} L^4}{EA}\left(\frac{L}{r}\right)^2 \hat{w}^3 + (\beta L)^4 \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} = 0 \quad (11)$$

Where $r = \sqrt{I/A}$ is the radius of gyration of the beam cross-section.

The solution of Equation (11) can be assumed as $\hat{w}(\hat{x},\hat{t}) = \phi(\hat{x})q(\hat{t})$ where $\phi(\hat{x})$ is the mode shape of the beam. For the clamped-clamped beam $\phi(\hat{x})$ is as follows[19]

$$\phi(\hat{x}) = \cosh(\beta_n \hat{x}) - \cos(\beta_n \hat{x}) - \frac{\cosh(\beta_n) - \cos(\beta_n)}{\sinh(\beta_n) - \sin(\beta_n)}(\sinh(\beta_n \hat{x}) - \sin(\beta_n \hat{x})) \quad (12)$$

Using the Galerkin's method and multiplying both sides of Equation (11) by $\phi(\hat{x})$ and integrating over the interval of [0,1] results in:

$$(\beta L)^4 \left(\int_0^1 \phi^2 d\hat{x}\right)\ddot{q} + \left[\left(\int_0^1 \phi\phi^{(4)} d\hat{x}\right) + \frac{F_{Th}}{EA}\left(\frac{L}{r}\right)^2\left(\int_0^1 \phi\phi'' d\hat{x}\right) + \frac{k_L L^2}{EA}\left(\frac{L}{r}\right)^2\left(\int_0^1 \phi^2 d\hat{x}\right)\right]q +$$
$$\left[-\frac{1}{2}\left(\frac{L}{r}\right)^2\left(\int_0^1 \phi'^2 d\hat{x}\right)\left(\int_0^1 \phi\phi'' d\hat{x}\right) + \frac{k_{NL} L^4}{EA}\left(\frac{L}{r}\right)^2\left(\int_0^1 \phi^4 d\hat{x}\right)\right]q^3 = 0 \quad (13)$$

After some mathematical manipulations one obtains:

$$\ddot{q} + \omega_0^2 q + \varepsilon q^3 = 0 \quad (14)$$

Where ω_0^2 and ε are:

$$\omega_0^2 = \frac{f_4 EI + f_2 L^2(EA\alpha\Delta T + hA\alpha^2 \Delta T^2) + f_1 k_L L^4}{f_1 EI \beta^4 L^4}$$

$$\varepsilon = \frac{-EA f_2 f_3 + 2f_5 k_{NL} L^4}{2f_1 EI \beta^4 L^2} \quad (15)$$

and

$$f_1 = \int_0^1 \phi^2 d\hat{x}, \quad f_2 = \int_0^1 \phi\phi'' d\hat{x}, \quad f_3 = \int_0^1 \phi'^2 d\hat{x}, \quad f_4 = \int_0^1 \phi\phi^{(4)} d\hat{x}, \quad f_5 = \int_0^1 \phi^4 d\hat{x} \quad (16)$$

Equation (14) is the differential equation of motion governing the nonlinear vibration of the beam. It is assumed that the initial conditions are:

$$q(0) = a_0 = \frac{w_{max}}{L}, \qquad \dot{q}(0) = 0 \quad (17)$$

Where w_{max} denotes the beam maximum deflection.

There are different methods to solve Equation (14). However, most of these methods don't result in a valid solution for the strongly nonlinear cases $(\varepsilon > 1)$. The nonlinear term coefficient, ε is dependent on the beam parameters as well as the boundary conditions. For the beam with the characteristics given in Table 1, ε is very large when compared with unity. For instance, for the clamped-clamped beam, $\varepsilon = 4130$. Therefore, the traditional perturbation methods which are based on the small parameter, ε, didn't lead to a valid expansion for the solution. In this paper the MHPM is used to obtain the beam response for $0 \leq \varepsilon < \infty$.

Table 1. The characteristics of the beam

3.9	b (mm)
6.4	h (mm)
485	L (mm)
7860	ρ (kg/m^3)
190	E (GPa)
1.2e-5	α (K^{-1})
-140E	h (GPa)
0.3	ν
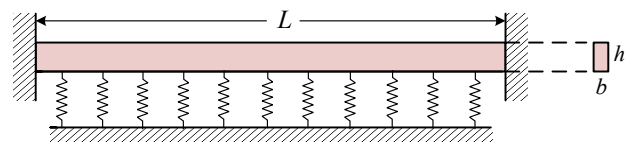
Figure 1. A clamped-clamped beam resting on a Pasternak foundation.

3. The system response with the MHPM

Perturbation methods have many limitations for solving and analyzing the behavior of strongly nonlinear systems. As described above to use perturbation techniques, the coefficient of the nonlinear term should be smaller than unity. In recent years, some methods are proposed to overcome this limitation. For example in[20], proposed

the modified Lindstedt-Poincare method. In this paper more accurate method called the Modified Homotopy Perturbation Method (MHPM) is applied to solve the mentioned strongly nonlinear equation. Based on the MHPM, the first order approximate solution is:

$$q(\tau) = a_0\cos\Omega\tau + \frac{\varepsilon a_0^3}{32\Omega^2}(\cos3\Omega\tau - \cos\Omega\tau) + o(\varepsilon^2) \quad (18)$$

Where, Ω is the first order nonlinear frequency and its value is as:

$$\Omega = \sqrt{\omega_0^2 + \frac{3}{4}\varepsilon a_0^2} \quad (19)$$

Consequently, the present first order approximate solution gives exactly the same result as the standard Lindstedt-Poincare method[21].

Also the second order approximate solution becomes as:

$$q(\tau) = a_0\cos\Omega\tau + \frac{\varepsilon a_0^3}{32\Omega^2}(\cos3\Omega\tau - \cos\Omega\tau) + \frac{\varepsilon^2 a_0^5}{1024\Omega^4}(\cos5\Omega\tau - \cos\Omega\tau) + o(\varepsilon^3) \quad (20)$$

Where, Ω is the second order nonlinear frequency as:

$$\Omega = \frac{1}{4}\sqrt{8\omega_0^2 + 6\varepsilon a_0^2 + \sqrt{64\omega_0^4 + 96\varepsilon a_0^2\omega_0^2 + 30\varepsilon^2 a_0^4}} \quad (21)$$

According to Equations (20) and (21), the period of free oscillations reads:

$$T = \frac{2\pi}{\Omega} \quad (22)$$

By qualitative analysis of conservative systems and integrating the level curve in phase plane for a given total energy level, Nayfeh obtained the exact value for the system period. After some manipulations he reached the system period as:

$$T_{ex} = \frac{4}{\sqrt{\omega_0^2 + \varepsilon a_0^2}}\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1 - \left(\frac{\varepsilon a_0^2}{2(\omega_0^2 + \varepsilon a_0^2)}\right)\sin^2 x}} \quad (23)$$

It is worth noting that the period of the first order approximate solution of the Variational Iteration Method (VIM) is[22]:

$$T = \frac{2\pi}{\sqrt{\left(10\omega_0^2 + 7\varepsilon a_0^2 + \sqrt{64\omega_0^4 + 104\varepsilon a_0^2\omega_0^2 + 49\varepsilon^2 a_0^4}\right)/18}} \quad (24)$$

The maximum relative error of the system period ($RE\,(\%)$) becomes as follows:

$$RE\,(\%) = \left|\lim_{\varepsilon\to\infty}\frac{T - T_{ex}}{T_{ex}}\right|\times100 \quad (25)$$

In present study, the beam characteristics are considered as given in Table 1.

4. Results and Discussion

According to Equations (21) to (25), the maximum relative error for the first and the second order approximation via the MHPM and the first order approximation through the VIM are 2.22%, 0.03% and 4.1%, respectively. As it is seen, the second order approximation of the MHPM is more accurate than the others. Also the VIM is too cumbersome for high order approximations and its accuracy is very low in compared with the MHPM. Table 2(a) shows the comparison of the no dimensional nonlinear natural frequency obtained through the present study, other methods and those reported in the literature for various values of vibrational amplitudes. Table 2(b) shows the comparison for the large values of vibrational amplitudes[23–27]. According to Table 2(a), there is an excellent agreement between the results obtained from the second order approximation of the MHPM and the exact solution. Moreover, to find the value of nonlinear frequency, the MHPM is more accurate and simpler solution than those available in the literature. From Table 2(b), it can be observed that there is a good agreement between the results obtained from the MHPM and the exact solution for the large values of vibrational amplitudes. In Figure 2 to Figure 4 indicate the responses of a clamped-clamped beam for various values of vibrational amplitudes. These figures reveal that the second order MHPM follows the RK45 for wide range of τ with a good accuracy. Figure 5 shows the responses of the beam for various values of increased temperature. As it can be seen from this figure, by raising the room temperature, the nonlinear frequency decreases because of strong preload in the beam. Figure 6 and Figure 7 illustrates the effect of the increased temperature, ΔT, on nonlinear fundamental natural frequency and the frequency ratio, respectively. As Figure 6 reveals, increasing the room temperature decreases the nonlinear natural frequency of the beam. Moreover, an increase in the vibration amplitude increases the fundamental natural frequency. This leads to the fact that by increasing the excitation amplitude, the nonlinearity dominates and the effect of increased temperature on the nonlinear

frequency becomes more significant. As it can be seen from Figure 7, with increasing the room temperature, the frequency ratio decreases. Although an increase in room temperature decreases both nonlinear natural frequency and frequency ratio, but as these figures show, its effects on decreasing the linear natural frequency is more than nonlinear one.

Table 2. Comparing the nondimensional nonlinear frequency obtained via:

(a) Different methods and those reported in the literature for the small values of εa_0^2

εa_0^2			Solving Method
2.25	1	0.25	
1.6257	1.3178	1.0892	Exact solution
1.6519	1.3277	1.0903	VIM
1.6394	1.3229	1.0897	First order MHPM
1.6256	1.3178	1.0892	Second order MHPM
1.6394	1.3229	1.0897	Azrar[23]
1.6393	1.3228	1.0897	HPM [24]
1.6394	1.3229	1.0897	Qaisi[25]
1.6394	1.3229	1.0897	Ritz Method[26]

(b) Different methods for the large values of εa_0^2

| Nondimensional Nonlinear Frequency				εa_0^2
Second order of present study	First order of present study	VIM	Exact solution	
1.3178	1.3229	1.3277	1.3178	1
2.8661	2.9155	2.9579	2.8667	10
8.5311	8.7178	8.8739	8.5336	100
26.8025	27.4044	27.9060	26.8107	1000
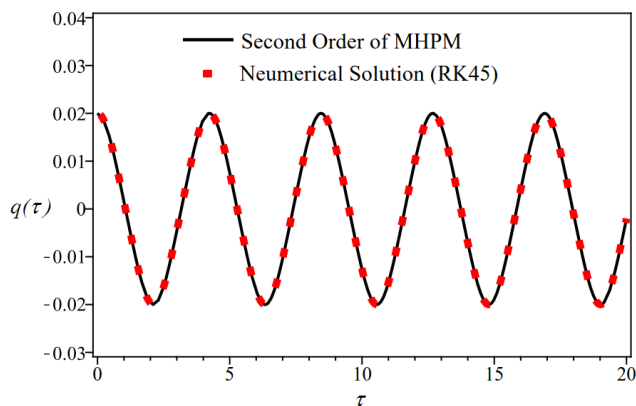
Figure 2. The responses obtained by the MHPM and RK45 method for the clamped-clamped beam.
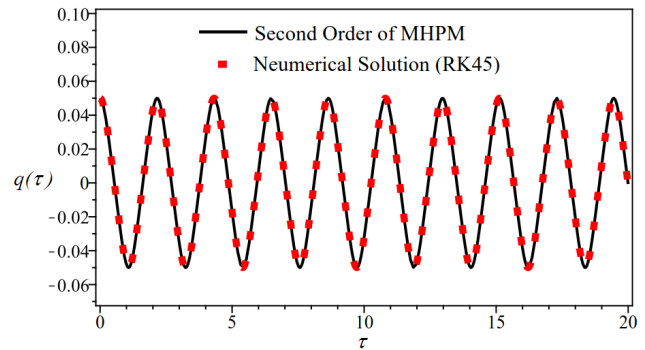
Figure 3. The responses obtained by the MHPM and RK45 method for the clamped-clamped beam.
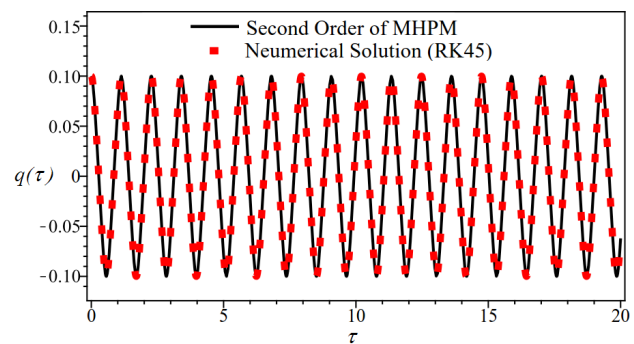
Figure 4. The responses obtained by the MHPM and RK45 method for the clamped-clamped beam.
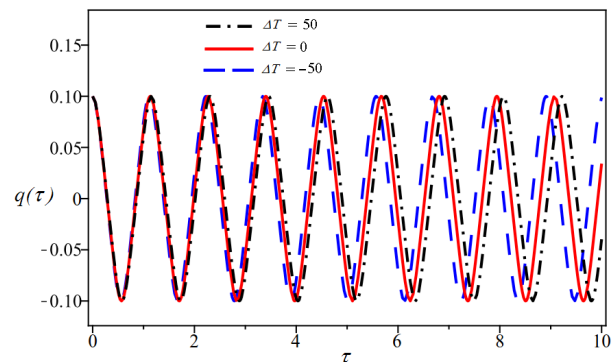

**Figure 5.** The responses of the beam by the MHPM for various values of increased temperature.

5. Conclusion

In this paper, large amplitude free vibration behavior of Euler-Bernoulli beam, corresponding to the first spatial mode and subjected to the nonlinear thermal loads is investigated. The boundary conditions are assumed doubly clamped and the beam was rested on a Pasternak foundation. Considering the effects of mid plane stretch-
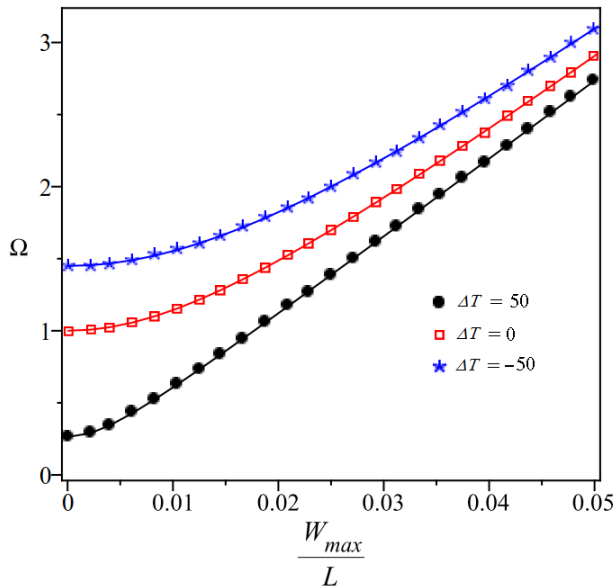
Figure 6. Variation of the nonlinear frequency, Ω, versus the amplitude for various values of increased temperature, ΔT .
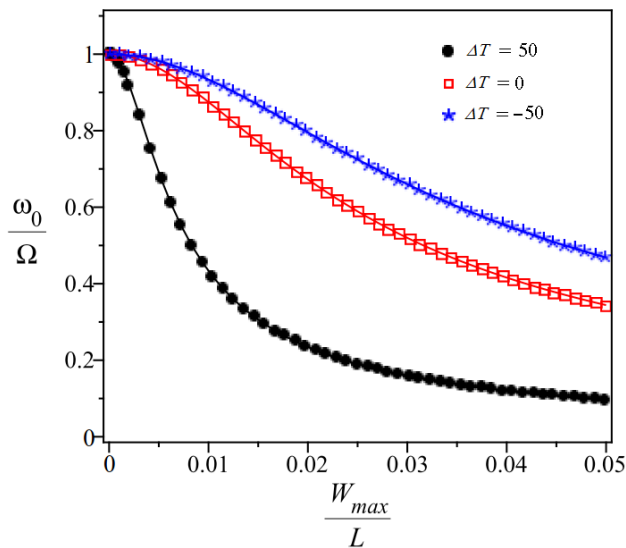
Figure 7. Variation of the frequency ratio, $\dfrac{\omega_0}{\Omega}$, versus the amplitude for various values of increased temperature, ΔT .

ing and implementing the Hamilton's principle and the Galerkin's method, the beam governing equation of motion is derived. Because of the large coefficient of the nonlinear term, the Modified Homotopy Perturbation Method (MHPM) is used to solve the obtained equation. Comparison among the frequencies obtained by the first and the second order MHPM, the VIM, the exact solu-

tion and those reported in the literature demonstrates the high accuracy of the second order MHPM. Moreover, the comparison between the time responses of the mentioned system obtained by the MHPM and the numerical technique shows the high accuracy of the second order MHPM. The effect of nonlinear thermal load on the system nonlinear vibration behavior is studied. The results show that although increasing the nonlinear thermal load coefficients decreases both linear and nonlinear frequency, but its effect on decreasing the linear natural frequency is more dominant. Also, the time responses show that by raising the room temperature, the nonlinear frequency decreases because of strong preload in the beam.

6. References

1. Krylov NN, Bogoliubov NN. Introduction to nonlinear mechanics. New Jersey: US: Princeton University Press; 1950. p. 106.
2. Bogoliubov NN, Mitropolski YA. Asymptotic methods in the theory of non-linear oscillations. 1st ed. New York: Gordon and Breach; 1961 Jan. p. 537.
3. Vainberg BR. Problems on asymptotic methods of non-stationary oscillations. 1975. Available from: http://iopscience.iop.org/article/10.1070/RM1975v030n02ABEH001406/pdf
4. Moghaddam NS, Skoracki R, Miller M, Elahinia M, Dean D. Three dimensional printing of stiffness-tuned, nitinol skeletal fixation hardware with an example of mandibular segmental defect repair. Procedia CIRP. 2016 Jul; 49:45–50.
5. Toward patient specific long lasting metallic implants for mandibular segmental defects. 2015. Available from: https://etd.ohiolink.edu/!etd.send_file?accession=toledo1438616258&disposition=inline
6. Nayfeh AH. Introduction to perturbation techniques.1st ed. NY: Wiley-VCH; 2011.
7. Murdock JA. Perturbations: Theory and methods. US: Society of Industrial Applied Mathematics; 1999.
8. Enhancement of bone implants by substituting nitinol for titanium (Ti-6Al-4V): A modeling comparison. 2014. Available from: http://proceedings.asmedigitalcollection.asme.org/proceeding.aspx?articleid=2022125
9. Esfahani SN, Andani MT, Moghaddam NS, Mirzaeifar R, Elahinia M. Independent tuning of stiffness and toughness of additively manufactured titanium-polymer composites: Simulation, fabrication and experimental studies. Journal of Materials Processing Technology. 2016 Dec; 238:22–9.
10. He JH. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering. 1999 Aug; 178(3-4):257–62.

11. He JH. A coupling method of a homotopy technique and a perturbation technique for non-linear problems. International Journal of Non-Linear Mechanics. 2000 Jan; 35(1):37–43.

12. He JH. Addendum: New interpretation of homotopy perturbation method. International Journal of Modern Physics B. 2006 Jun; 20(18):2561–8.

13. He JH. A new perturbation technique which is also valid for large parameters. Journal of Sound and Vibration. 2000 Feb; 229(5):1257–63.

14. A theoretical and experimental investigation on large amplitude free vibration behavior of a pretensioned beam with clamped–clamped ends using modified homotopy perturbation method. 2015. Available from: http://pic.sagepub.com/content/early/2015/04/14/0954406215580663.abstract

15. Malekzadeh P, Vosoughi A. DQM large amplitude vibration of composite beams on nonlinear elastic foundations with restrained edges. Communications in Nonlinear Science and Numerical Simulation. 2009; 14(3):906–15.

16. Alexander J, Augustine BS. Free vibration and damping characteristics of GFRP and BFRP laminated composites at various boundary conditions. Indian Journal of Science and Technology. 2015; 8(12):1–7.

17. Akbarzade M, Khan Y. Dynamic model of large amplitude non-linear oscillations arising in the structural engineering: Analytical solutions. Mathematical and Computer Modelling. 2012 Feb; 55(3-4):480–9.

18. Qian Q, Wang L, Ni Q. Instability of simply supported pipes conveying fluid under thermal loads. Mechanics Research Communications. 2009 Apr; 36(3):413–7.

19. Sedighi HM, Reza A, Zare J. Study on the frequency amplitude relation of beam vibration. International Journal of Physical Sciences. 2011 Dec; 6(36):8051–6.

20. Cheung Y, Chen S, Lau S. A modified Lindstedt-Poincare method for certain strongly non-linear oscillators. International Journal of Non-Linear Mechanics. 1991; 26(3-4):367–78.

21. He JH. Modified Lindstedt–Poincare methods for some strongly non-linear oscillations: Part I: Expansion of a constant. International Journal of Non-Linear Mechanics. 2002 Mar; 37(2):309–14.

22. Variational iteration method to duffing equation. 1999. Available from: http://en.cnki.com.cn/Article_en/CJFDTOTAL-JSWL902.001.htm

23. Azrar L, Benamar R, White R. Semi-analytical approach to the non-linear dynamic response problem of S–S and C–C beams at large vibration amplitudes part I: General theory and application to the single mode approach to free and forced vibration analysis. Journal of Sound and Vibration. 1999 Jul; 224(2):183–207.

24. Pirbodaghi T, Ahmadian M, Fesanghary M. On the homotopy analysis method for non-linear vibration of beams. Mechanics Research Communications. 2009 Mar; 36(2):143–8.

25. Qaisi MI. Application of the harmonic balance principle to the non-linear free vibration of beams. Applied Acoustics.1993; 40(2):141–51.

26. Lewandowski R. Application of the Ritz method to the analysis of non-linear free vibrations of beams. Journal of Sound and Vibration. 1987; 114(1):91–101.

27. A numerical simulation of the effect of using porous superelastic Nitinol and stiff Titanium fixation hardware on the bone remodeling. 2016. Available from: http://proceedings.spiedigitallibrary.org/proceeding.aspx?articleid=2523508