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Cannon Fired Ball with Relative Velocity

Kausar Jahan[1], S. Koteswara Rao[2] and Rudra Pratap Das[3].

[1]Department of ECE, Vignan's Institute of Information Technology, Visakhapatnam – 530046, Andhra Pradesh, India; kausar.465@gmail.com
[2]Department of ECE, K L University, Vaddeswaram – 522502, Guntur District, Andhra Pradesh, India; rao.sk9@gmail.com
[3]Department of ECE, Vignan's Institute of Information Technology, Visakhapatnam – 530046, Andhra Pradesh, India; drrudrapratapdas@yahoo.com

Abstract

Radar is assumed to be tracking a cannon launched projectile travelling in a 2 dimensional drag environment. Extended Kalman Filter (EKF) is explored for the estimation of position and velocity of the projectile. Simulation is carried out using MATLAB and the results are presented. Finally, based on the results, EKF is recommended for tracking the projectile.

Keywords: Ballistics, Estimation, Extended Kalman Filter, Prediction, Relative velocity

1. Introduction

During flight gravity and drag have a major impact on the path of the projectile and are to be considered when predicting how the projectile will travel. Gravity imparts a downward acceleration causing the projectile to drop from the line of sight. Drag decelerates the projectile with a force proportional to the square of the velocity. As we are taking the velocities in both X and Y directions considering the drag, the velocity in one direction will have impact on the velocity in the other direction. So, while obtaining the acceleration in one direction, instead of taking direct square of the velocity we need to take the vector form of the velocity, i.e. the velocity in a particular direction multiplied by the relative velocity.

2. Mathematical Modelling

Consider a simple example in which radar tracks a cannon-launched projectile travelling in two-dimension (with drag environment). After the projectile is launched at an initial velocity, gravity as well as air resistance acts on the projectile. So we can say that in the downrange or x-direction there is acceleration due to air drag and in the y-direction or altitude there is acceleration of gravity and air drag. Considering the relative velocity

$$v = \sqrt{v_x^2 + v_y^2} \tag{1}$$

$$a_x = \frac{0.0034e^{-x_T/22000} g v_x \sqrt{v_x^2 + v_y^2}}{2\beta} \tag{2}$$

$$a_y = \frac{0.0034e^{-y_T/22000} g v_y \sqrt{v_x^2 + v_y^2}}{2\beta} - g \tag{3}$$

We can express the location of the cannon-launched projectile (i.e., x_T, y_T) in terms of the radar range (r) and angle (α) as

$$x_T = r\cos\alpha + x_R \tag{4}$$

$$y_T = r\sin\alpha + y_R \tag{5}$$

$$\alpha = \tan^{-1}\left(\frac{y_T - y_R}{x_T - x_r}\right) \tag{6}$$

$$r = \sqrt{(y_T - y_R)^2 + (x_T - x_R)^2} \tag{7}$$

*Author for correspondence

Where x_T and y_T are the co-ordinates of target location and x_R and y_R are the co-ordinates of radar location.

The states for the cannon launched projectile tracking model can be chosen as projectile location and velocity in the downrange or x direction and projectile location and velocity in the altitude or y direction and the ballistic co-efficient (β) as it is unknown. Thus, the proposed states are given by

$$X_k = \begin{bmatrix} x_T \\ v_x \\ y_T \\ v_y \\ \beta \end{bmatrix} \quad (8)$$

Therefore, when the preceding Cartesian states are chosen the state-space differential equation describing projectile motion becomes

$$\begin{pmatrix} \Delta v_x \\ \Delta a_x \\ \Delta v_y \\ \Delta a_y \\ \Delta\dot{\beta} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial v_x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial v_y} & \frac{\partial v_x}{\partial \beta} \\ \frac{\partial a_x}{\partial x} & \frac{\partial a_x}{\partial v_x} & \frac{\partial a_x}{\partial y} & \frac{\partial a_x}{\partial v_y} & \frac{\partial a_x}{\partial \beta} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial v_x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial v_y} & \frac{\partial v_y}{\partial \beta} \\ \frac{\partial a_y}{\partial x} & \frac{\partial a_y}{\partial v_x} & \frac{\partial a_y}{\partial y} & \frac{\partial a_y}{\partial v_y} & \frac{\partial a_y}{\partial \beta} \\ \frac{\partial \dot{\beta}}{\partial x} & \frac{\partial \dot{\beta}}{\partial v_x} & \frac{\partial \dot{\beta}}{\partial y} & \frac{\partial \dot{\beta}}{\partial v_y} & \frac{\partial \dot{\beta}}{\partial \beta} \end{pmatrix} \begin{pmatrix} \Delta x_T \\ \Delta v_x \\ \Delta y_T \\ \Delta v_y \\ \Delta\dagger \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ n_p \\ 0 \\ n_p \\ n_p \end{pmatrix} \quad (9)$$

In equation 9 gravity g is not a state that has to be estimated but is assumed to be known in advance. We have also added process noise n_p to the acceleration portion of the equations as protection for effects that may not be considered by the Kalman filter. Thus the systems dynamic matrix is given by

$$D = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial v_x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial v_y} & \frac{\partial v_x}{\partial \beta} \\ \frac{\partial a_x}{\partial x} & \frac{\partial a_x}{\partial v_x} & \frac{\partial a_x}{\partial y} & \frac{\partial a_x}{\partial v_y} & \frac{\partial a_x}{\partial \beta} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial v_x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial v_y} & \frac{\partial v_y}{\partial \beta} \\ \frac{\partial a_y}{\partial x} & \frac{\partial a_y}{\partial v_x} & \frac{\partial a_y}{\partial y} & \frac{\partial a_y}{\partial v_y} & \frac{\partial a_y}{\partial \beta} \\ \frac{\partial \dot{\beta}}{\partial x} & \frac{\partial \dot{\beta}}{\partial v_x} & \frac{\partial \dot{\beta}}{\partial y} & \frac{\partial \dot{\beta}}{\partial v_y} & \frac{\partial \dot{\beta}}{\partial \beta} \end{pmatrix} \quad (10)$$

The fundamental matrix φ(t) for a time in variant system is given by Taylor series approximation is given by

$$\varphi(t) = I + Dt + \frac{(Dt)^2}{2!} + \cdots + \frac{(Dt)^n}{n!} + \cdots \quad (11)$$

As the fundamental matrix will only be used in the Riccati equations but not used in state propagation, we will take only the first two terms of the Taylor series expansion

$$\varphi(t) \approx I + Dt = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ d_{21}t & 1+d_{22}t & 0 & d_{24}t & d_{25}t \\ 0 & 0 & 1 & t & 0 \\ 0 & d_{42}t & d_{43}t & 1+d_{44}t & d_{45}t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (12)$$

Where

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & d_{24} & d_{25} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & d_{42} & d_{43} & d_{44} & d_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (13)$$

$$d_{21} = \frac{-0.0034e^{-x_T/22000} g v_x \sqrt{v_x^2 + v_y^2}}{44000\beta}$$

$$d_{22} = \frac{0.0034e^{-x_T/22000} g \left(2v_x^2 + v_y^2\right)}{\beta}$$

$$d_{24} = \frac{0.0034e^{-x_T/22000} g v_x v_y}{\beta\sqrt{v_x^2 + v_y^2}}$$

$$d_{25} = \frac{-0.0034e^{-x_T/22000} g v_x \sqrt{v_x^2 + v_y^2}}{2\beta^2}$$

$$d_{42} = \frac{0.0034e^{-y_T/22000} g v_x v_y}{\beta\sqrt{v_x^2 + v_y^2}}$$

$$d_{43} = \frac{-0.0034e^{-y_T/22000} g v_y \sqrt{v_x^2 + v_y^2}}{44000\beta}$$

$$d_{44} = \frac{0.0034e^{-y_T/22000} g\left(v_x^2 + 2v_y^2\right)}{\beta\sqrt{v_x^2 + v_y^2}}$$

$$d_{45} = \frac{-0.0034e^{-y_T/22000} g v_y \sqrt{v_x^2 + v_y^2}}{2\beta^2}$$

∴ The discrete fundamental matrix can be found by substituting T_s for t and is given by

$$\varphi_k = \begin{pmatrix} 1 & T_s & 0 & 0 & 0 \\ d_{21}T_s & 1 + d_{22}T_s & 0 & d_{24}T_s & d_{25}T_s \\ 0 & 0 & 1 & T_s & 0 \\ 0 & d_{42}T_s & d_{43}T_s & 1 + d_{44}T_s & d_{45}T_s \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (14)$$

As we have chosen states to be Cartesian, the radar measurements r and α will automatically be nonlinear functions of those states. Therefore, we must write the linearized measurement equation as

$$\begin{pmatrix} \Delta\alpha^* \\ \Delta r^* \end{pmatrix} = \begin{pmatrix} \frac{\partial\alpha}{\partial x_T} & \frac{\partial\alpha}{\partial y_T} & \frac{\partial\alpha}{\partial y_T} & \frac{\partial\alpha}{\partial v_y} & \frac{\partial\alpha}{\partial\beta} \\ \frac{\partial r}{\partial x_T} & \frac{\partial r}{\partial v_x} & \frac{\partial r}{\partial y_T} & \frac{\partial r}{\partial v_y} & \frac{\partial r}{\partial\beta} \end{pmatrix} \begin{pmatrix} \Delta x_T \\ \Delta v_x \\ \Delta y_T \\ \Delta v_y \\ \Delta\beta \end{pmatrix} + \begin{pmatrix} n_\alpha \\ n_r \end{pmatrix} \quad (15)$$

where n_α and n_r represent the measurement noise on angle and range, respectively. Because the angle from the radar to the projectile is given by

$$\alpha = \tan^{-1}\left(\frac{y_T - y_R}{x_T - x_R}\right) \quad (16)$$

The five partial derivatives of the angle with respect to each of the states are computed as

$$\frac{\partial\alpha}{\partial x_T} = \frac{-\left(y_T - y_R\right)}{r^2} \quad (17)$$

$$\frac{\partial\alpha}{\partial x_x} = 0 \quad (18)$$

$$\frac{\partial\alpha}{\partial y_T} = \frac{\left(x_T - x_R\right)}{r^2} \quad (19)$$

$$\frac{\partial\alpha}{\partial v_y} = 0 \quad (20)$$

$$\frac{\partial\alpha}{\partial\beta} = 0 \quad (21)$$

The five partial derivatives of the range with respect to each of the states are computed as

$$\frac{\partial r}{\partial x_T} = \frac{\left(x_T - x_R\right)}{r} \quad (22)$$

$$\frac{\partial r}{\partial v_x} = 0 \quad (23)$$

$$\frac{\partial r}{\partial y_T} = \frac{\left(y_T - y_R\right)}{r} \quad (24)$$

$$\frac{\partial r}{\partial v_y} = 0 \quad (25)$$

$$\frac{\partial r}{\partial\beta} = 0 \quad (26)$$

The linearized measurement matrix is given by

$$M = \begin{pmatrix} \frac{\partial\alpha}{\partial x_T} & \frac{\partial\alpha}{\partial v_x} & \frac{\partial\alpha}{\partial y_T} & \frac{\partial\alpha}{\partial v_y} & \frac{\partial\alpha}{\partial\beta} \\ \frac{\partial r}{\partial x_T} & \frac{\partial r}{\partial v_x} & \frac{\partial r}{\partial y_T} & \frac{\partial r}{\partial v_y} & \frac{\partial r}{\partial\beta} \end{pmatrix} \quad (27)$$

Substituting equations (17) – (26) in equation (27) gives the linearized measurement matrix as

$$M = \begin{pmatrix} \frac{-\left(y_T - y_R\right)}{r^2} & 0 & \frac{\left(x_T - x_R\right)}{r^2} & 0 & 0 \\ \frac{\left(x_T - x_R\right)}{r} & 0 & \frac{\left(y_T - y_R\right)}{r} & 0 & 0 \end{pmatrix} \quad (28)$$

For this problem it is assumed that we know where the radar is so that x_R and y_R are known and do not have to be estimated. The states required for the discrete linearized measurement matrix will be based on the projected state estimate or

$$M_k = \begin{pmatrix} \frac{-\left(\bar{y}_{T_k} - y_R\right)}{\bar{r}_k^2} & 0 & \frac{\left(\bar{x}_{T_k} - x_R\right)}{\bar{r}^2} & 0 & 0 \\ \frac{\left(\bar{x}_{T_k} - x_R\right)}{\bar{r}_k} & 0 & \frac{\left(\bar{y}_{T_k} - y_R\right)}{\bar{r}} & 0 & 0 \end{pmatrix} \quad (29)$$

The discrete measurement noise matrix is given by

$$R_k = E\left(n_k n_k^T\right)$$

$$R_k = \begin{pmatrix} \sigma_\alpha^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix} \quad (30)$$

where σ_α^2 and σ_r^2 are the variances of the angle noise and range noise measurements, respectively. Similarly the continuous process-noise matrix is given by

$$Q = E[ww^T]$$

$$Q(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_s & 0 \\ 0 & 0 & 0 & 0 & \phi_s \end{pmatrix} \quad (31)$$

where ϕ_s is the spectral density of the white noise. Sources assumed to be on the downrange and altitude accelerations acting on the projectile. The discrete process-noise matrix can be derived from the continuous process-noise matrix according to

$$Q_k = \int_0^{T_s} \varphi(\tau) Q \varphi^T(\tau) d\tau \quad (32)$$

Substitution of the equations (12) and (31) into the equation (32) yields

$$Q_k = \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} \end{pmatrix} \quad (33)$$

Where

$$q_{11} = \frac{T_s^3 \phi_s}{3}$$

$$q_{12} = \left(\frac{T_s^3 \phi_s}{2} + \frac{d_{22} T_s^3 \phi_s}{3} \right)$$

$$q_{13} = q_{31} = 0$$

$$q_{14} = q_{41} = \frac{T_s^3 \phi_s d_{42}}{3}$$

$$q_{15} = q_{51} = 0$$

$$q_{21} = q_{12}$$

$$q_{22} = \left(T_s + d_{22} T_s^2 + \frac{d_{22}^2 T_s^3}{3} \right) \phi_s + \frac{d_{24}^2 T_s^3 \phi_s}{3} + \frac{d_{25}^2 T_s^3 \phi_s}{3}$$

$$q_{23} = q_{32} = \frac{d_{24} T_s^3 \phi_s}{3}$$

$$q_{24} = q_{42} = \frac{d_{42} T_s^2 \phi_s}{2} + \frac{d_{22} d_{42} T_s^3 \phi_s}{3} + \frac{d_{24} T_s^2 \phi_s}{2} + \frac{d_{44} d_{24}}{3}$$

$$q_{25} = q_{52} = \frac{d_{25} T_s^2 \phi_s}{2}$$

$$q_{33} = \frac{T_s^3 \phi_s}{3}$$

$$q_{34} = q_{43} = \frac{T_s^3 \phi_s}{2} + \frac{d_{44} T_s^3 \phi_s}{3}$$

$$q_{35} = q_{53} = 0$$

$$q_{44} = \frac{d_{42}^2 T_s^3 \phi_s}{3} + \left(T_s + \frac{d_{44} T_s^2}{2} + \frac{d_{44}^2 T_s^3}{3} \right)^2 \phi_s + \frac{d_{45}^2 T_s^3 \phi_s}{3}$$

$$q_{45} = q_{54} = \frac{d_{45} T_s^2 \phi_s}{2}$$

$$q_{55} = T_s \phi_s$$

To propagate the states from the present sampling time to the next sampling time, we use the fundamental matrix with Runge–Kutta second order differential equation.

For the linear filtering problem the real world was represented by the state-space equation

$$\dot{X}_k = DX_s + Gu + w \quad (34)$$

where **G** is a matrix multiplying a known disturbance or control vector **u** that does not have to be estimated. We can show that the discrete linear Kalman-filtering equation is given by

$$\hat{x}_k = \varphi_k \hat{x}_{k-1} + G_k u_{k-1} + K_k \left(z_k - H \varphi_k \hat{x}_{k-1} - H G_k u_{k-1} \right)$$

Where K_k is the Kalman gain matrix, z_k is measurement equation in discrete form, H is measurement matrix and G_k is given by

$$G_k = \int_0^{T_s} \phi(\tau) G d\tau$$

For this problem

$$G = Gu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g \\ 0 \end{pmatrix}$$

Therefore $$G_k = \begin{pmatrix} 0 \\ -\dfrac{d_{24}T_s^2 g}{2} \\ -g\dfrac{T_s^2}{2} \\ -T_s\left[1+\dfrac{d_{44}T_s}{2}\right]g \\ 0 \end{pmatrix}$$

The Kalman gains are computed from the Riccati equations which are given as

$$C_k = \phi_k P_{k\text{-}1}\phi_k^T + Q_k$$

$$K_k = C_k M^T\left(MC_k M^T + R_k\right)^{-1}$$

$$P_k = \left(I - K_k M\right)C_k$$

where C_k is the covariance matrix representing the errors in the state estimates before an update, P_k is a covariance matrix representing the errors in the state estimates after an update.

If we forget about the gain times the residual portion of the filtering equation, we can see that the projected state is simply

$$\bar{x}_k = \phi_k\hat{x}_{k-1} + G_k u_{k-1}$$

and our projected state is determined from

$$\tilde{x}_k = \begin{pmatrix} 1 & \tau & 0 & 0 & 0 \\ d_{21}\tau & 1+d_{22}\tau & 0 & 0 & d_{22}\tau \\ 0 & 0 & 1 & \tau & 0 \\ 0 & 0 & 1+d_{43}\tau & 1+d_{44}\tau & d_{45}\tau \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\tilde{x}_{k-1} + \begin{pmatrix} 0 \\ 0 \\ -g\dfrac{T_s^2}{2} \\ -T_s\left[1+\dfrac{d_{44}T_s}{2}\right] \\ 0 \end{pmatrix}u_{k-1} \qquad (35)$$

3. Simulation Results

The path of a cannon fired projectile is not exactly a parabola in real time which is obtained when using relative velocity which can be seen from Figure 1. The downrange and altitude also differ while using relative velocity.

From Figure 2, we can see that the error in estimation of downrange is low (of the order 10^5) when using relative velocity than the one without relative velocity (of order 10^{152}).

From Figure 3, we can see that simulation result of the error in estimation of altitude is much nearer to the theoretical bounds when using relative velocity than the one without relative velocity.

The error in the estimation of downrange velocity is also more within the theoretical bounds when using relative velocity rather than not using the relative velocity. The same can be observed from Figure 4.

Error in the estimation of altitude velocity is low and within theoretical bounds in the case of using relative

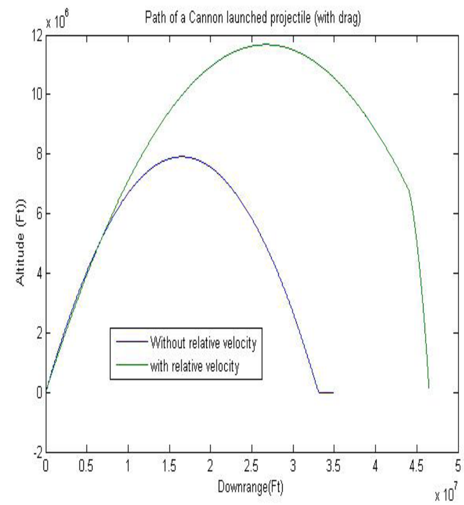

Figure 1. Path of a cannon fired projectile.

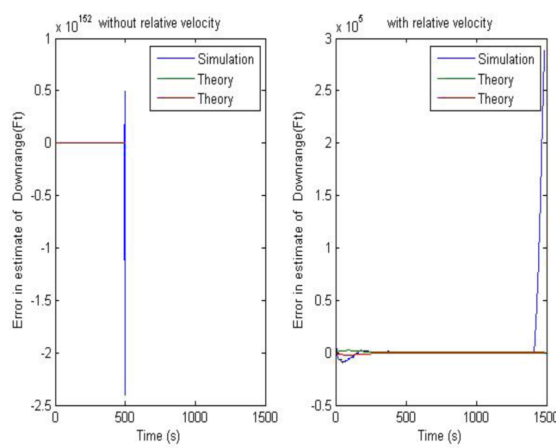

Figure 2. Error in estimation of downrange.

velocity while the error in estimation of altitude velocity is high though within the theoretical bounds in case of not using the relative velocity, which can be observed from Figure 5.

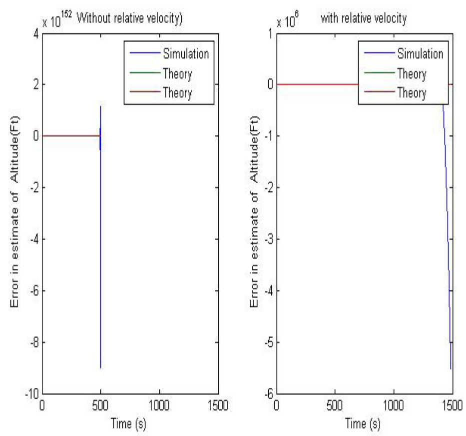

Figure 3. Error in the estimation of altitude.

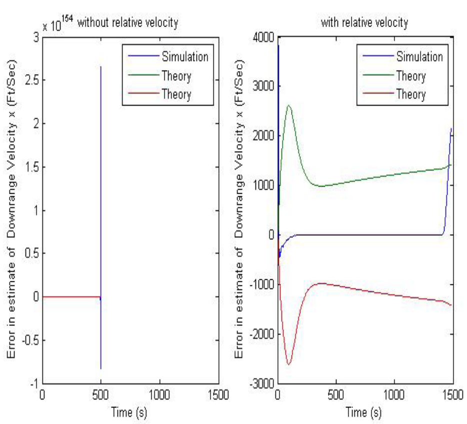

Figure 4. Error in estimation of downrange velocity.

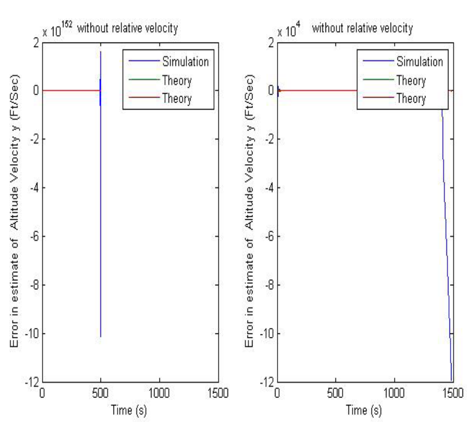

Figure 5. Error in estimation of altitude velocity.

4. Conclusion

From the simulation results it is observed that the estimation of down range and altitude differ a lot when relative velocity is considered. As we are taking drag into consideration velocity vector has to be taken instead of taking velocities in individual directions.

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