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Congruences and External Direct Sum of LA-modules

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Abstract

In this paper we study a new algebraic structure namely left almost module (LA-module in short).We extend the notion of congruences to LA-modules which is defined in[1] for semigroups. We show that every homomorphism defines a congruence relation on LA-modules and prove analogues of isomorphism theorems. We also define external direct sum of LA-modules and show that the internal direct sum of LA-submodules is isomorphic to the external direct sum of those LA-submodules.

Keywords: Congruences, External Direct Sums, Internal Direct Sums, LA-modules, LA-module Homomorphism, LA-submodules

1. Introduction

M. A. Kazim and M. Naseeruddin[2] introduced the notion of left almost semigroups (LA-semigroups). A groupoid S with binary operation '*' is said to be a left almost

Semigroup if it satisfies the left invertive law i.e. $(a * b) * c = (c * b) * a \;\forall\, a, b, c \in \boldsymbol{S}$.

LA-semigroup is also known as an Abel-Grassmann's groupoid (AG-groupoid)[3]. In[4], medial and paramedial groupoid ware initiated. A medial is a groupoid $\boldsymbol{S}$ satisfying the medial law: $(ab)\,(cd) = (ac)\,(bd)$ while a paramedial is a groupoid $\boldsymbol{S}$ satisfying the paramedial law: $(ab)\,(cd) = (db)\,(ca) \;\forall\, a, b, c, d \in \boldsymbol{S}$. They proved that AG-groupoid $\boldsymbol{S}$ always satisfies the medial law: $(ab)\,(cd) = (ac)\,(bd)$ [Lemma 1.1(i)] while an AG-groupoid $\boldsymbol{S}$ with left identity e satisfies paramedial law: $(ab)\,(cd) = (db)\,(ca)$ [Lemma 1.2(ii)]. In[5], the author proved that, an AG-groupoid $\boldsymbol{S}$ with left identity e also satisfies $a\,(bc) = b(ac) \;\forall\, a, b, c \in \boldsymbol{S}$ [Lemma 4]. Basically an LA-semigroup is the generalization of a commutative semigroup. In[6], M.S. Kamran extended the concept of LA-semigroup to a left almost group (LA-group) which are non-associative structures. A groupoid $\boldsymbol{G}$ with the binary operation '*' a binary operation is said to be an LA-group if the following conditions are satisfied: (i) There exists an element $e \in \boldsymbol{G}$ such that $e * a = a \;\forall\, a \in \boldsymbol{G}$, (ii) For $a \in \boldsymbol{G}$ there exists $a^{-1} \in \boldsymbol{G}$ such that $a^{-1} * a = a * a^{-1} = e$, (iii) Left invertive law holds in $\boldsymbol{G}$. An LA-group is basically the generalization of a commutative group. LA-groups have interesting resemblance to commutative groups. In[7], Q. Mushtaq, M. S. Kamran proved most useful results about the said structure. In[8], S.M. Yusuf extended the notion of LA-groups to left almost rings (LA-rings), the non-associative structures with two binary operations '+' and '·'. A left almost ring is a non-empty set $\boldsymbol{R}$ together with two binary operations '+' and '·' satisfying the following:

(i) $(\boldsymbol{R}, +)$ is an LA-group, (ii) $(\boldsymbol{R},\cdot)$ is an LA-semigroup, (iii) Both left and right distributive laws holds. In[9], T. Shah and I. Rehman introduced the concept of LA-modules over LA-rings. Basically the conditions of LA-modules are close to that of modules which are abelian groups.

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In[10], T. Shah, M. Raees and G. Ali extended the structure to its substructure and obtained some useful results. In the same paper, they defined LA-module homeomorphisms in a similar way as that of modules. The terms endomorphism, monomorphism, epimorphism, isomorphism and automorphism can be defined in the same way. In the said paper they proved some useful results. In particular they proved first, second and third isomorphism theorems. They also defined internal direct sum of LA-submodules. In this study we give the concept of congruences on LA-modules and show that every homomorphism defines a congruence relation on LA-modules. We also define external direct sum of LA-modules and show that internal direct sum is isomorphic to external direct sum.

2. Preliminaries

In this section we give some basic definitions and theorems which have been taken from[9,10]. We shall use these results in later sections.

2.1 Definition[9]

An LA-group ($\boldsymbol{M}$, +) is said to be an LA-module over an LA-ring ($\boldsymbol{R}$, +, $\cdot$) with left identity 1, if the mapping $\boldsymbol{R}\times\boldsymbol{M}\to\boldsymbol{M}$ defined as $(r, m \to rm \in \boldsymbol{M}$, where $m \in \boldsymbol{M}$ and $r \in \boldsymbol{R}$, satisfies the following conditions:

- $r\,(m_1+m_2) = rm_1 + rm_2$,
- $(r_1+r_2)\,m = r_1m+r_2m$,
- $r_1\,(r_2m) = r_2\,(r_1m)$,
- $1\cdot m = m$, $\forall\, r, r_1, r_2 \in \boldsymbol{R}$ and $m, m_1, m_2 \in \boldsymbol{M}$[9].

It is obvious from the above definition that, every LA-ring $\boldsymbol{R}$ with left identity 1 is an LA-module over itself. We are now going to give a non-trivial example of an LA-module which has been taken from the source[9].

2.2 Example

Let $\boldsymbol{S}$ be a Commutative semigroup and ($\boldsymbol{R}$, +, $\cdot$) an LA-ring. Then,

$\boldsymbol{R}[\boldsymbol{S}] = \{\sum_{finite} r_i s_i : r_i \in \boldsymbol{R}$ and $s_i \in \boldsymbol{S}\}$ under the mapping $\boldsymbol{R} \times \boldsymbol{R}[\boldsymbol{S}] \to \boldsymbol{R}[\boldsymbol{S}]$ defined by

$$\left(\sum_{i=1}^{n} r_i s_i\right) \mapsto \sum_{i=1}^{n} (rr_i) s_i$$ is an LA-module.

2.3 Definition

An LA-subgroup $\boldsymbol{N}$ of an LA-module $\boldsymbol{M}$ over an LA-ring $\boldsymbol{R}$ is said to be an LA-submodule over $\boldsymbol{R}$, if $\boldsymbol{R}\,\boldsymbol{N} \subseteq \boldsymbol{N}$. In other words $r\,n \in \boldsymbol{N}$ for all $r \in \boldsymbol{R}$ and $n \in \boldsymbol{N}$.

2.4 Theorem

[Theorem 2 in10]. If $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are two LA-submodules of an LA-module $\boldsymbol{M}$ over an LA-ring $\boldsymbol{R}$, then $\boldsymbol{M}_1 \cap \boldsymbol{M}_2$ is an LA-submodule of $\boldsymbol{M}$.

2.5 Definition

Let $\boldsymbol{M}$ be an LA-module over an LA-ring $\boldsymbol{R}$ with left identity 1 and $\boldsymbol{N}$ an LA-submodule of $\boldsymbol{M}$. We define the quotient LA-module $\boldsymbol{M/N} = \{m + \boldsymbol{N}\!: m \in \boldsymbol{M}\}$.That is, $\boldsymbol{M/N}$ is the set of all cosets of $\boldsymbol{M}$ in $\boldsymbol{N}$.

2.6 Definition

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two LA-modules over an LA-ring $\boldsymbol{R}$ with left identity 1.A mapping $\phi\!: \boldsymbol{M} \to \boldsymbol{N}$ is said to be an LA-module homomorphism if, $\forall\, r \in \boldsymbol{R}$ and $m_1, m_2 \in \boldsymbol{M}$ the following conditions are satisfied:

- $(m_1 + m_2)\,\phi = (m_1)\,\phi + (m_2)\,\phi$,
- $(r\,m_1)\,\phi = r\,(m_1)\,\phi$.

2.7 Definition

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two LA-modules over an LA-ring $\boldsymbol{R}$ with left identity 1. Suppose ϕ: $\boldsymbol{M} \to \boldsymbol{N}$ is an LA-module homomorphism. Then kernel of ϕ is defined as:

$$\boldsymbol{ker}\phi = \{m \in \boldsymbol{M} : (m)\,\phi = 0\}.$$

2.8 Theorem

[Theorem 7 in 10]. Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two LA-modules, and ϕ: $\boldsymbol{M} \to \boldsymbol{N}$ an LA-module Epimorphism. Then **,** $\boldsymbol{M/ker}\phi \cong \boldsymbol{N}$. More generally if ϕ: $\boldsymbol{M} \to \boldsymbol{N}$ is an LA-module homomorphism then $\boldsymbol{M/ker}\phi \cong \boldsymbol{Img}\phi$.

The above theorem is said to be First Isomorphism Theorem for LA-modules.

2.9 Definition

Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be LA-submodules of an LA-module $\boldsymbol{M}$. Then $\boldsymbol{M}$ is called internal direct sum of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, if every element $m \in \boldsymbol{M}$ can be written in one and only one way as

$m = m_1 + m_2$, where $m_1 \in \boldsymbol{M_1}$ and $m_2 \in \boldsymbol{M_2}$. It is denoted symbolically as $\boldsymbol{M} = \boldsymbol{M_1} \oplus \boldsymbol{M_2}$.

The following result gives equivalence conditions for internal direct sums.

2.10 Theorem

[Theorem 10 in 10]. Let $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ be LA-submodules of an LA-module $\boldsymbol{M}$. Then $\boldsymbol{M}$ is the internal direct sum of $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ if and only if.

- $\boldsymbol{M} = \boldsymbol{M_1} + \boldsymbol{M_2}$
- $\boldsymbol{M_1} \cap \boldsymbol{M_2} = \{0\}$.

The following result is modified form of the Exercise on page 178 of[11] which is true for modules. Here we prove the modified form for LA-modules.

2.11 Theorem

Let $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ be two LA-submodules of an LA-module $\boldsymbol{M}$ such that $\boldsymbol{M} = \boldsymbol{M_1} \oplus \boldsymbol{M_2}$. Then

- ϕ: $\boldsymbol{M_1} \oplus \boldsymbol{M_2} \to \boldsymbol{M_2}$ defined by $(m_1 + m_2)\,\phi = m_2$ for all $m_1 + m_2 \in \boldsymbol{M_1} \oplus \boldsymbol{M_2}$ is an LA-module epimorphism and $\boldsymbol{M_1} \oplus \boldsymbol{M_2}/\boldsymbol{ker}\phi \cong \boldsymbol{M_2}$.
- ϕ: $\boldsymbol{M_1} \oplus \boldsymbol{M_2} \to \boldsymbol{M_1}$ defined by $(m_1 + m_2)\,\phi = m_1$ for all $m_1 + m_2 \in \boldsymbol{M_1} \oplus \boldsymbol{M_2}$ is an LA-module homomorphism and $\boldsymbol{M_1} \oplus \boldsymbol{M_2}/\boldsymbol{M_2} \cong \boldsymbol{Img}\phi$.

Proof: *(i)* To show that ϕ: $\boldsymbol{M_1} \oplus \boldsymbol{M_2} \to \boldsymbol{M_2}$ is an epimorphism. We first show that ϕ is well defined. Let $m_1 + m_2$, $m'_1 + m'_2 \in \boldsymbol{M_2} \oplus \boldsymbol{M_1}$, be such that

$m_1 + m_2 = m'_1 + m'_2$

$\Rightarrow (m_1 + m_2) - m_2 = (m'_1 + m'_2) - m_2$

so $(-m_2 + m_2) + m_1 = (-m_2 + m'_2) + m'_1$ ($\because$ by left invertive law)

$\Rightarrow m_1 - m'_1 = ((-m_2 + m'_2) + m'_1) - m'_1$

$= (-m'_1 + m'_1) + (-m_2 + m'_2)$ ($\because$by left invertive law) Thus $m_1 - m'_1 = -m_2 + m'_2 \in \boldsymbol{M_1} \cap \boldsymbol{M_2} = \{0\}$

$\Rightarrow \quad -m_2 + m_1 = 0$

$\Rightarrow \quad m_2 = m'_2$

$\Rightarrow \quad (m_1 + m_2)\,\phi = (m'_1 + m'_2)\,\phi$.

Now $((m_1 + m_2) + (m'_1 + m'_2))\,\phi = ((m_1 + m'_1) + (m_2 + m'_2))\,\phi$ ($\because$ by medial law)

Thus $\quad = m_2 + m'_2$

$\quad = (m_1 + m_2)\,\phi + (m'_1 + m'_2)\,\phi$.

Let, $r \in \boldsymbol{R}$ then, $\quad (r\,(m_1 + m_2))\,\phi = (r\,m_1 + r\,m_2)\,\phi$

$\quad = r\,m_2$

Thus $= r\,(m_1 + m_2)\,\phi$.

Hence ϕ is an LA-module homomorphism.

Let $m_2 \in \boldsymbol{M_2}$, then $m_2 = 0 + m_2 \in \boldsymbol{M_1} + \boldsymbol{M_2}$. Thus $(m_2)\,\phi = m_2$ which implies that ϕ is onto. Hence ϕ is an LA-module epimorphism. Thus by first isomorphism theorem we have $\boldsymbol{M}/\boldsymbol{ker}\phi = \boldsymbol{M_2}$.

- Is well defined follows from the above part *(i)*. To show that ϕ: $\boldsymbol{M_1} \oplus \boldsymbol{M_2} \to \boldsymbol{M_1}$ is a Homomorphism let $m_1 + m_2$, $m'_1 + m'_2 \in \boldsymbol{M_1} \oplus \boldsymbol{M_2}$, $r \in \boldsymbol{R}$ then
  $((m_1 + m_2) + (m'_1 + m'_2))\,\phi = ((m_1 + m'_1) + (m_2 + m'_2))\,\phi$ ($\because$ by medial law)

Thus $\quad = m_1 + m'_1$

$\quad = (m_1 + m_2)\,\phi + (m'_1 + m'_2)\,\phi$.

Now $\quad (r\,(m_1 + m_2))\,\phi = (r\,m_1 + r\,m_2)\,\phi$

$\quad = r\,m_1$

Thus $\quad = r\,(m_1 + m_2)\,\phi$.

Hence ϕ is an LA-module homomorphism.

Thus by first isomorphism theorem we have $\boldsymbol{M_1} \oplus \boldsymbol{M_2}/\boldsymbol{ker}\phi \cong \boldsymbol{Img}\phi$.

We show that $\boldsymbol{ker}\phi = \boldsymbol{M_2}$. Now let, $m_1 + m_2 \in \boldsymbol{ker}\phi$ then $(m_1 + m_2)\,\phi = 0$, but since $(m_1 + m_2)\,\phi = m_1$ thus $m_1 = 0$.

Now $m_1 + m_2 = 0 + m_2 \in \boldsymbol{M_2}$ which implies that $m_1 + m_2 \in \boldsymbol{M_2}$ implies $\boldsymbol{ker}\phi \subseteq \boldsymbol{M_2}$.

Now let, $m_2 \in \boldsymbol{M_2}$ then $m_2 = 0 + m_2 \in \boldsymbol{M_1} + \boldsymbol{M_2}$. Therefore $(m_2)\,\phi = 0$, so $m_2 \in \boldsymbol{ker}\phi$. It follows that $\boldsymbol{M_2} \subseteq \boldsymbol{ker}\phi$. Thus, $\boldsymbol{M_2} = \boldsymbol{ker}\phi$. Hence $\boldsymbol{M_1} \oplus \boldsymbol{M_2}/\boldsymbol{M_2} \cong \boldsymbol{Img}\phi$.

3. Congruences

In this section we discuss congruences on LA-modules. Also we shall prove analogues of isomorphism theorems using the concept of congruence's. The idea comes from the book 1 in which the author has done similar calculation for semigroups.

3.1 Definition

Let $(\boldsymbol{M}, +)$ be an LA-module over an LA-ring $(\boldsymbol{R}, + \cdot)$ with left identity 1.A relation ρ on the set $\boldsymbol{M}$ is said to be compatible, if for all $m_1, m_2, m_3, m_4 \in \boldsymbol{M}$ and for all $r \in \boldsymbol{R}$ such that (m_1, m_2) and $(m_3, m_4) \in \rho \Rightarrow (m_1 + m_2, m_3 + m_4) \in \rho$ and $(r\,m_1, r\,m_2) \in \rho$.

A compatible equivalence relation is said to be a congruence relation.

3.2 Example

Consider an LA-ring of order 7, with addition and multiplication are defined in following tables.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	6	0	1	2	3	4	5
2	5	6	0	1	2	3	4
3	4	5	6	0	1	2	3
4	3	4	5	6	0	1	2
5	2	3	4	5	6	0	1
6	1	2	3	4	5	6	0

·	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Where 0 is the left additive identity and 1 is the left multiplicative identity. According to remark after the definition of an LA-module the above defined LA-ring is an LA-module over itself. Let $\rho = \{(a, b): a = b\}$ be a relation on the above defined LA-module. Then one can easily verify that ρ is a congruence relation.

We are now going to prove a result in which we show that every LA-module Homomorphism defines a congruence relation on LA-modules.

3.3 Theorem

If ϕ is an LA-module Homomorphism from an LA-module $\boldsymbol{M}$ to an LA-module $\boldsymbol{N}$. Then ϕ defines a congruence relation ρ on $\boldsymbol{M}$ given by $(m_1, m_2) \in \rho$ if and only if $(m_1)\phi = (m_2)\phi$, for all $m_1, m_2 \in \boldsymbol{M}$.

Proof: First we show that this relation is an equivalence relation. Since for all $m \in \boldsymbol{M}$,

$(m)\phi = (m)\phi$, so $(m, m) \in \rho$ which implies that ρ is reflexive. Let $m_1, m_2 \in \boldsymbol{M}$ such that $(m_1, m_2) \in \rho$. Then $(m_1)\phi = (m_2)\phi$ which implies that $(m_2)\phi = (m_1)\phi$. Thus $(m_2, m_1) \in \rho$ and so ρ is symmetric. Let $m_1, m_2, m_3 \in \boldsymbol{M}$ such that $(m_1, m_2) \in \rho$ and $(m_2, m_3) \in \rho$ then $(m_1)\phi = (m_2)\phi$ and $(m_2)\phi = (m_3)\phi$. So, $(m_1)\phi = (m_3)\phi$ which implies that $(m_1, m_3) \in \rho$. Thus ρ is transitive. It follows that ρ is an equivalence relation. Next let $m_1, m_2, m_3, m_4 \in \boldsymbol{M}$ such that $(m_1, m_2) \in \rho$ and $(m_3, m_4) \in \rho$ then $(m_1)\phi = (m_2)\phi$ and $(m_3)\phi = (m_4)\phi$. Since ϕ is an LA-module Homomorphism so, $(m_1 + m_3)\phi = (m_1)\phi + (m_3)\phi = (m_2)\phi + (m_4)\phi$ and so, $(m_1 + m_3)\phi = (m_2 + m_4)\phi$ which implies that $(m_1 + m_3, m_2 + m_4) \in \rho$. Also for all $r \in \boldsymbol{R}$, $r(m_1)\phi = r(m_2)\phi$. Since ϕ is an LA-module homomorphism so, $(r\,m_1)\phi = (r\,m_2)\phi$ which implies that $(r\,m_1, r m_2) \in \rho$, and hence ρ is compatible. It follows that ρ is congruence.

3.4 Definition

Let $\boldsymbol{M}$ be an LA-module over an LA-ring $(\boldsymbol{R}, +, \cdot)$ with left identity 1 and ρ a congruence relation on $\boldsymbol{M}$. We define $\boldsymbol{M}/\rho = \{(m)^\rho : m \in \boldsymbol{M}\}$. That is, $\boldsymbol{M}/\rho$ consists of all equivalence classes corresponding to the elements of $\boldsymbol{M}$.

Suppose ρ is a congruence relation on $\boldsymbol{M}$. Then we can make $\boldsymbol{M}/\rho$ to be an LA-module over the same LA-ring $(\boldsymbol{R}, +, \cdot)$ with left identity 1 by defining the following binary operations:

$(m_1)\rho + (m_2)\rho = (m_1 + m_2)\rho$ and $(r\,m)\rho = r\,(m)\rho$ for all $m, m_1, m_2 \in \boldsymbol{M}$ and $r \in \boldsymbol{R}$.

Let $m_1, m_2, m_3, m_4 \in \boldsymbol{M}$ be such that $(m_1)^\rho = (m_2)^\rho$ and $(m_3)\rho = (m_4)\rho$ then $(m_1, m_2) \in \rho$ and

$(m_3, m_4) \in \rho$. Since, ρ is a congruence relation so, $(m_1 + m_3, m_2 + m_4) \in \rho$ and for all $r \in \boldsymbol{R}$, $(r\,m_1, r\,m_2) \in \rho$.

It follows that $(m_1 + m_3)\rho = (m_2 + m_4)\rho$ and $(r\,m_1)\rho = (r\,m_2)\rho$. Thus the operations are well defined.

Now let $m_1, m_2, m_3, m_4 \in \boldsymbol{M}$ and $r \in \boldsymbol{R}$ such that $(m_1)\rho$, $(m_2)\rho$, $(m_3)\rho$, $(m_4)\rho \in \boldsymbol{M}/\rho$, then

$$
\begin{aligned}
((m_1)\rho + (m_2)\rho) + (m_3)\rho &= ((m_1 + m_2)\rho) + (m_3)\rho \\
&= ((m_1 + m_2) + m_3)\rho \\
&= ((m_3 + m_2) + m_1)\rho \ (\because \text{by left invertive law}) \\
&= (m_3 + m_2)\rho + (m_1)\rho \\
&= ((m_3)\rho + (m_2)\rho) + (m_1)\rho
\end{aligned}
$$

Thus, $\boldsymbol{M}/\rho$ satisfies the left invertive law.

Now since left additive identity, $0 \in \boldsymbol{M}$, therefore, $(0)\rho \in \boldsymbol{M}/\rho$. So for $(m)\rho \in \boldsymbol{M}/\rho$, we have

$(0)\rho + (m)\rho = (0 + m)\rho = (m)\rho$. Thus, $(0)\rho$ is the left additive identity of $\boldsymbol{M}/\rho$. Also, since

$m \in \boldsymbol{M}$ implies $-m \in \boldsymbol{M}$, therefore, $(-m)\rho \in \boldsymbol{M}/\rho$. So for $(m)\rho \in \boldsymbol{M}/\rho$ we have $(-m)\rho\,(m)\rho$

$= (-m + m)\rho = (0)\rho$ and $(m)\rho + (-m)\rho = (m - m)\rho = (0)\rho$. It follows that $(-m)\rho$ is the additive inverse of $\boldsymbol{M}/\rho$.

Thus $\boldsymbol{M}/\rho$ is an LA-group.

Next let $r, r_1, r_2 \in \boldsymbol{R}$. Then

$$
\begin{aligned}
\text{(i)}\ r\,((m_1)\rho + (m_2)\rho) &= r\,((m_1 + m_2)\rho \\
&= (r\,(m_1 + m_2))\rho \\
&= (r\,m_1 + r\,m_2)\rho \\
&= (r\,m_1)\rho + (r\,m_2)\rho \\
&= r\,(m_1)\rho + r\,(m_2)\rho
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii)}\ (r_1 + r_2)\,(m)\rho &= ((r_1 + r_2)\,m)\rho \\
&= (r_1 m + r_2 m)\rho \\
&= (r_1 m)\rho + (r_2 m)\rho \\
&= r_1\,(m)\rho + r_2\,(m)\rho
\end{aligned}
$$

$$
\begin{aligned}
\text{(iii)}\ r_1\,(r_2\,(m)\rho) &= r_1\,(r_2 m)\rho \\
&= (r_1\,(r_2 m))\rho \\
&= (r_2\,(r_1 m))\rho \\
&= r_2\,(r_1 m)\rho \\
&= r_2\,(r_1\,(m)\rho)
\end{aligned}
$$

(iv) $1\cdot (m)\rho = (1\cdot m)\rho$
$= (m)\rho.$

Thus $\boldsymbol{M}/\rho$ is an LA-module.

We are now going to prove analogues of isomorphism theorems. Theorem 3.5 is analogues of first isomorphism theorem, Theorem 3.6 is analogues of second isomorphism theorem and Theorem 3.7 is analogues of third isomorphism theorem. These results have been taking from1 which is true for Semigroups.

3.5 Theorem

If ρ is a congruence relation on an LA-module $\boldsymbol{M}$. Then $\boldsymbol{M}/\rho$ is an LA-module with respect to the binary operation defined as:

$(m_1)\rho + (m_2)\rho = (m_1 + m_2)\rho$ *and* $r\,(m)\rho = (r\,m)\rho$ for all $m, m_1, m_2 \in \boldsymbol{M}$ and $r \in \boldsymbol{R}$.

The mapping $\rho^{\#}: \boldsymbol{M} \to \boldsymbol{M}/\rho$ defined by $(m)^{\#}\rho = (m)\rho$ for all $m \in \boldsymbol{M}$ is an LA-module Epimorphism.

If ϕ: $\boldsymbol{M} \to \boldsymbol{N}$ is a LA-module homomorphism where $\boldsymbol{M}$ and $\boldsymbol{N}$ are LA-modules. Then the relation

$\boldsymbol{ker}\phi = \{(m_1, m_2) \in \boldsymbol{M} \times \boldsymbol{M}: (m_1)\,\phi = (m_2)\,\phi\}$

is a congruence relation on $\boldsymbol{M}$ and there is an LA-module Monomorphism $\psi : \boldsymbol{M}/\boldsymbol{ker}\phi \to \boldsymbol{N}$ such that $\boldsymbol{Img}\phi = \boldsymbol{Img}\phi$ and the diagram commutes.

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "(ker\phi)^{\#}"'] & & N \\ & M/ker\phi \arrow[ur, "\psi"'] & \end{tikzcd}$$

Proof: The first part of the theorem follows from the above discussion. Now let $m, m_1, m_2 \in \boldsymbol{M}$ and $r \in \boldsymbol{R}$, then

$(m_1 + m_2)\rho^{\#} = (m_1 + m_2)\rho$
$= (m_1)\rho + (m_2)\rho$
$= (m_1)\rho^{\#} + (m_2)\rho^{\#}$

and $(r\,m)\rho^{\#} = (r\,m)\rho$
$= r\,(m)\rho$
$= r\,(m\,\rho^{\#}.$

Thus $\rho^{\#}$ is an LA-module homomorphism. Clearly $\rho^{\#}$ is onto. Hence $\rho^{\#}$ is an LA-module epimorphism. From Theorem 3.3 the relation $\boldsymbol{ker}\phi$ is an equivalence relation.

Now define a mapping ψ: $\boldsymbol{M}/\boldsymbol{ker}\phi \to \boldsymbol{N}$ by $((m)\,\boldsymbol{ker\phi})\,\psi = (m)\,\phi$ for all $(m)\,\boldsymbol{ker\phi} \in \boldsymbol{M}/\boldsymbol{ker}\phi$. Then ψ is well defined and one-one.

Let $(m_1)\boldsymbol{ker\phi}$, $(m_2)\,\boldsymbol{ker\phi} \in \boldsymbol{M}/\boldsymbol{ker}\phi$ such that $(m_1)\,\boldsymbol{ker\phi} = (m_2)\,\boldsymbol{ker\phi} \Leftrightarrow (m_1, m_2) \in \boldsymbol{ker\phi} \Leftrightarrow (m_1)\,\phi = (m_2)\,\phi \Leftrightarrow ((m_1)\,\boldsymbol{ker\phi})\,\psi = ((m_2)\,\boldsymbol{ker\phi})\,\psi.$

Now

$((m_1)\,\boldsymbol{ker\phi} + (m_2)\,\boldsymbol{ker\phi})\,\psi = ((m_1 + m_2)\,\boldsymbol{ker\phi})\,\psi$
$= (m_1 + m_2)\,\phi$
$= (m_1)\,\phi + (m_1)\,\phi$
$= ((m_1)\,ker\phi)\,\psi + ((m_2)\boldsymbol{ker\phi})\,\psi$

Now let, $(m)\,\boldsymbol{ker\phi} \in \boldsymbol{M}/\boldsymbol{ker}\phi$ and $r \in \boldsymbol{R}$ then

$(r\,(m)\,\boldsymbol{ker\phi})\,\psi = ((r\,m)\boldsymbol{ker\phi})\,\psi$
$= (r\,m)\,\phi$
$= r\,(m)\,\phi$
$= r\,((m)\,\boldsymbol{ker\phi})\,\psi.$

Showing that ψ is an LA-module homomorphism.

Hence ψ is a LA-module monomorphism. It is obvious that $\boldsymbol{Img}\phi = \boldsymbol{Img}\psi$.

Now from the definition it is clear that $(m)\,[\boldsymbol{ker\phi}\#]\,\psi = [(m)\,\boldsymbol{ker\phi}\#]\,\psi = ((m)\,\boldsymbol{ker\phi})\,\psi = (m)\,\phi.$

Thus the diagram commutes.

3.6 Theorem

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be LA-modules over the same LA-ring ($\boldsymbol{R}$, +, $\cdot$) with left identity 1 and ϕ: $\boldsymbol{M} \to \boldsymbol{N}$ a LA-module Homomorphism. Suppose ρ is a congruence relation on LA-module $\boldsymbol{M}$ such that $\rho \subseteq \boldsymbol{ker}\phi$. Then there exists a unique LA-module homomorphism ψ: $\boldsymbol{M}/\rho \to \boldsymbol{N}$ such that $\boldsymbol{Img}\psi = \boldsymbol{Img}\phi$ and the diagram commutes.

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & N \\ & M/\rho \arrow[ur, "\rho^{\#}"'] & \end{tikzcd}$$

Proof: Define ψ: $\boldsymbol{M}/\rho \to \boldsymbol{N}$ by $((m)\rho)\,\psi = (m)\,\phi$ for all $(m)\rho \in \boldsymbol{M}/\rho$. Let $(m_1)\rho$, $(m_2)\rho \in \boldsymbol{M}/\rho$ such that $(m_1)\rho = (m_2)\rho$ then $(m_1, m_2) \in \rho \subseteq \boldsymbol{ker}\phi$ which implies that $(m_1, m_2) \in \boldsymbol{ker}\phi$. Thus

$(m_1)\,\phi = (m_2)\,\phi$. It follows that ψ is well-defined. Now let $(m_1)\rho$, $(m_2)\rho \in \boldsymbol{M}/\rho$, then

$((m_1)\rho + (m_2)\rho)\,\psi = ((m_1 + m_2)\rho)\,\psi$
$= (m_1 + m_2)\,\phi$
$= (m_1)\,\phi + (m_2)\,\phi$
$= ((m_1)\rho)\,\psi + ((m_2)\rho)\,\psi.$

Also for $r \in \boldsymbol{R}$ we have, $(r\,(m_1)\rho)\,\psi = ((r\,m_1)\rho)\,\psi$
$= (r\,m_1)\,\phi$
$= r\,(m_1)\,\phi$
$= r\,((m_1))\,\psi.$
Thus ψ is a LA-module homomorphism.
It is clear that $\boldsymbol{Img}\phi = \boldsymbol{Img}\psi$.
Now $(m)\,(\rho^{\#})\,\psi = ((m)\rho^{\#})\,\psi = ((m)\rho)\,\psi = (m)\,\phi$. That is the diagram commutes.
Now let ψ_1: $\boldsymbol{M}/\rho \to \boldsymbol{N}$ be another LA-module Homomorphism such that $(\rho^{\#})\,\psi_1 = \phi$. Let $m \in \boldsymbol{M}$ then $(m)\,(\rho^{\#})\,\psi_1 = ((m)\rho^{\#})\,\psi_1 = (m)\,\phi = (m)\,(\rho^{\#})\,\psi$ which implies that $((m))\,\psi_1 = ((m))\,\psi$.
Thus $\psi = \psi_1$. Hence the LA-module Homomorphism ψ is unique.

3.7 Theorem

Let ρ and ρ be congruence relations on LA-module $\boldsymbol{M}$ such that $\rho \subseteq \rho$. Then $\rho/\rho = \{((m_1)^{\rho}, (m_2)^{\rho}) \in \boldsymbol{M}/\rho \times \boldsymbol{M}/\rho : (m_1, m_2) \in \rho\}$ is a congruence relation on $\boldsymbol{M}/\rho$ and $\boldsymbol{M}/\rho\,/\sigma/\rho \cong \boldsymbol{M}/\sigma$.

Proof: First we show that the relation σ/ρ is a congruence relation on $\boldsymbol{M}/\rho$. Since $(m, m) \in \sigma$ for all $m \in \boldsymbol{M}$, thus $((m)\rho, (m)\rho) \in \sigma/\rho$ which implies that σ/ρ is reflexive.

Let $m_1, m_2 \in \boldsymbol{M}$ be such that $((m_1)\rho, (m_2)\rho) \in \sigma/\rho$, then $(m_1, m_2) \in \sigma$. Since σ is Symmetric so

$(m_2, m_1) \in \rho$ And so, $((m_2)\rho, (m_1)\rho) \in \sigma/\rho$. Hence σ/ρ is Symmetric. Now let $m_1, m_2, m_3 \in \boldsymbol{M}$ such that $((m_1)\rho, (m_2)\rho) \in \sigma/\rho$ and $((m_2)\rho, (m_3)\rho) \in \sigma/\rho$, then $(m_1, m_2) \in \sigma$ and $(m_2, m_3) \in \sigma$. Since σ is Transitive therefore, $(m_1, m_3) \in \sigma$. Therefore $((m_1)\rho, (m_2)\rho) \in \sigma/\rho$. Hence σ/ρ is Transitive. So σ/ρ is an equivalence relation.

Now let $m_1, m_2, m_3, m_4 \in \boldsymbol{M}$ and $r \in R$ be such that, $((m_1)\rho, (m_2)\rho)$ and $((m_3)\rho, (m_4)\rho) \in \sigma/\rho$. Then (m_1, m_2) and $(m_3, m_4) \in \sigma$ but ρ is compatible therefore, $(m_1 + m_3, m_2 + m_4) \in \sigma$ and

$(r\,m_1, r\,m_2) \in \sigma$. Thus $((m_1 + m_3)\rho, (m_2 + m_4)\rho) \in \sigma/\rho$ and $((r\,m_1)\rho, (r\,m_2)\rho) \in \sigma/\rho$. Thus σ/ρ

is compatible. Hence σ/ρ is a congruence relation on $\boldsymbol{M}/\rho$. Next, define a mapping

ϕ: $\boldsymbol{M}/\rho \to \boldsymbol{M}/\sigma$ by $((m)\rho)\,\phi = \sigma(m)$.

Let $(m_1)\rho, (m_2)\rho \in \boldsymbol{M}/\rho$ such that $(m_1)\rho = (m_2)\rho$ then $(m_1, m_2)^{\in\rho} \subseteq \sigma$ which implies that
$(m_1, m_2) \in \sigma$ thus $(m_1)^{\sigma} = (m_2)^{\sigma}$ which implies that $((m_1)\rho)\,\phi = ((m_2)\rho)\,\phi$. It follows that ϕ is well-defined. Now let, $(m_1)\rho, (m_2)\rho \in \boldsymbol{M}/\rho$ then
$((m_1)\rho + (m_2)\rho)\,\phi = ((m_1 + m_2)\rho)\,\phi$
$= (m_1 + m_2)^{\sigma}$
$= (m_1)^{\sigma} + (m_2)^{\sigma}$
$= ((m_1))\,\phi + ((m_2))\,\phi.$
Also for $r \in \boldsymbol{R}$ we have, $(r\,(m_1))\,\phi = ((r\,m_1)\rho)\,\phi$
$= (r\,m_1)^{\sigma}$
$= r\,(m_1)^{\sigma}$
$= r\,((m_1))\,\phi.$

Thus by Theorem 3.5 there is an LA-module Monomorphism ψ: $\boldsymbol{M}/\rho\,/\boldsymbol{ker}\phi \to \boldsymbol{M}/\sigma$ defined by,

$((m)\rho)\,\boldsymbol{ker\phi})\,\psi = (m)^{\sigma}$. Clearly it is onto, because for $(m)^{\sigma} \in \boldsymbol{M}/\rho$ there exists

$((m)\rho)\,\boldsymbol{ker\phi} \in \boldsymbol{M}/\rho/\boldsymbol{ker}\phi$ such that, $(((m)\rho)\,\boldsymbol{ker\phi})\,\psi = (m)\sigma$. Hence $\boldsymbol{M}/\rho\,/\,\boldsymbol{ker}\phi \cong \boldsymbol{M}/\sigma$.

Now, $\boldsymbol{ker}\phi = \{((m_1)\rho, (m_2)\rho) \in \boldsymbol{M}/\rho \times \boldsymbol{M}/\rho : ((m_1)\rho)\,\phi = ((m_2)\rho)\,\phi\}$
$= \{((m_1)\rho, (m_2)\rho) \in \boldsymbol{M}/\rho \times \boldsymbol{M}/\rho : (m_1)^{\sigma} = (m_2)\rho\,\}$
$= \{((m_1)\rho, (m_2)\rho) \in \boldsymbol{M}/\rho \times \boldsymbol{M}\,/\rho : (m_1\,m_2) \in \sigma\}$
$= \sigma/\rho$
Hence $\boldsymbol{M}/\rho\,/\sigma/\rho \cong \boldsymbol{M}/\sigma$.

4. External Direct Sum

In[10] the authors have defined internal direct sum of LA-submodules $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of an LA-module $\boldsymbol{M}$. In this section we define external direct sum of LA-modules. We show that internal and external direct sums are isomorphic and prove a result which is based on external direct sum.

4.1 Definition

Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be LA-modules over the same LA-ring $(\boldsymbol{R}, +, \cdot)$ with left identity 1. Then we can define addition and scalar multiplication on the set $\boldsymbol{M}_1 \times \boldsymbol{M}_2$ as follows:

$$(m_1, m_2) + (m'_1, m'_2) = (m_1 + m'_1, m_2 + m'_2)$$

And

$$r\,(m_1, m_2) = (r\,m_1, r\,m_2).$$

For all $(m_1, m_2), (m'_1, m'_2) \in \boldsymbol{M}1 \times \boldsymbol{M}2$ and $r \in \boldsymbol{R}$. In other words and addition and scalar multiplication are defined coordinate wise. According to the above binary operation $\boldsymbol{M}1 \times \boldsymbol{M}2$ become an LA-module over the same LA-ring $\boldsymbol{R}$ which we call external direct sum of $\boldsymbol{M}1$ and $\boldsymbol{M}2$. It is denoted by $\boldsymbol{M}1 \otimes \boldsymbol{M}2$.

The following result shows that the external and internal direct sums are isomorphic.

4.2 Theorem

Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be two LA-submodules of an LA-module $\boldsymbol{M}$ such that

$\boldsymbol{M} = \boldsymbol{M}_1 \oplus \boldsymbol{M}_2$ then $\boldsymbol{M} \cong \boldsymbol{M}_1 \otimes \boldsymbol{M}_2$.

Proof: Define a mapping $\phi : \boldsymbol{M} \to \boldsymbol{M_1} \otimes \boldsymbol{M_2}$ by $(m_1 + m_2)\, \phi = (m_1, m_2)$ for all $m_1 + m_2 \in \boldsymbol{M}$

Let m1 + m_2, $m'_1 + m'_2 \in \boldsymbol{M}$ be such that

$m_1 + m_2 = m'_1 + m'_2$

$\Rightarrow \quad (m_1 + m_2) - m_2 = (m'_1 + m'_2) - m_2$

$\Rightarrow \quad (-m_2 + m_2) + m_1 = (- m_2 + m'_2) + m'_1$ ($\because$ by left invertive law)

$\Rightarrow \quad m_1 = (- m_2 + m'_2) + m'_1$

$\Rightarrow \quad m_1 - m'_1 = ((-m_2 + m'_2) + m'_1) - m'_1$

$\Rightarrow \quad m_1 - m'_1 = (- m'_1 + m'_1) + (-m_2 + m'_2)$ ($\because$ by left invertive law)

$\Rightarrow \quad m_1 - m'_1 = - m_2 + m'_2 \in \boldsymbol{M_1} \cap \boldsymbol{M_2}$

By Theorem 2.10 we have $\boldsymbol{M_1} \cap \boldsymbol{M_2} = \{0\}$ so, $m_1 - m'_1 = 0$ and $- m_2 + m'_2 = 0$. Thus, $m_1 = m'_1$ and $m_2 = m'_2$ Therefore, $(m_1, m_2) = (m'_1, m_2')$. Hence, $(m_1 + m_2)\, \phi = (m'_1 + m'_2)\, \phi$. It follows that ϕ is well-defined. Also it is obvious from the above discussion that ϕ is one-one.

Let $m_1 + m_2$, $m'_1 + m'_2 \in \boldsymbol{M_1} \oplus \boldsymbol{M_2}$ then

$((m_1 + m_2) + (m'_1 + m'_2))\, \phi = ((m_1 + m'_1) + (m_2 + m'_2))\, \phi$ ($\because$ by medial law)

$= (m_1 + m'_1, m_2 + m'_2)$

$= (m_1, m_2) + (m'_1, m'_2)$

$= (m_1 + m_2)\, \phi + (m'_1 + m'_2)\, \phi$.

Now let, $r \in R$ then $(r\,(m_1 + m_2))\, \phi = (r\, m_1 + r\, m_2)\, \phi$

$= (r\, m_1, r\, m_2)$

$= r\,(m_1, m_2)$

$= r\,(m_1 + m_2)\, \phi$.

Thus ϕ is an LA-module homomorphism. Clearly ϕ is onto; because for $(m_1, m_2) \in \boldsymbol{M_1} \otimes \boldsymbol{M_2}$ there exists $m_1 + m_2 \in \boldsymbol{M_1} \oplus \boldsymbol{M_2}$ such that $(m_1 + m_2)\, \phi = (m_1, m_2)$. Hence $\boldsymbol{M} \cong \boldsymbol{M_1} \otimes \boldsymbol{M_2}$.

We are now going to prove a result which has been taken from[12] which is true for ideals in a ring. Here we prove it for LA-modules.

4.3 Theorem

Let $\boldsymbol{M}$ be an LA-module over an LA-ring $(\boldsymbol{R}, +, \cdot)$ with left identity 1. Suppose $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ be two LA-submodules of $\boldsymbol{M}$. Then $\boldsymbol{M/M_1} \cap \boldsymbol{M_2} \cong \boldsymbol{M/M_1} \otimes \boldsymbol{M/M_2}$.

Proof: Define a mapping $\phi : \boldsymbol{M} \to \boldsymbol{M/M}1 \otimes \boldsymbol{M/M_2}$ by $(m)\, \phi = (m + \boldsymbol{M}1, m + \boldsymbol{M}2)$. Then ϕ is well defined. Let m, $m' \in M$ be such that $m = m'$ then $m + \boldsymbol{M}1 = m' + M_1$ and $m + \boldsymbol{M}2 = m' + \boldsymbol{M}2$ which implies that $(m + \boldsymbol{M}1, m + \boldsymbol{M}2) = (m' + \boldsymbol{M}1, m + \boldsymbol{M}2)$ thus, $(m)\, \phi = (m')\, \phi$.

Now let $m, m' \in \boldsymbol{M}$, then, $(m + m')\, \phi = ((m + m') + \boldsymbol{M}1, (m + m') + \boldsymbol{M}2)$

$= ((m + \boldsymbol{M}1) + (m' + \boldsymbol{M}1), (m + \boldsymbol{M}2) + (m' + \boldsymbol{M}2))$

$= (m + \boldsymbol{M}1, m + \boldsymbol{M}2) + (m' + \boldsymbol{M}1, m' + \boldsymbol{M}2)$

$= (m)\, \phi + (m')\, \phi$.

Now for all $r \in \boldsymbol{R}$ we have, $(r\, m)\, \phi = (r\, m + \boldsymbol{M}1, r\, m + \boldsymbol{M}2)$

$= r\,(m + \boldsymbol{M}1, m + \boldsymbol{M}2)$

$= r\,(m)\, \phi$.

Thus ϕ is LA-module homomorphism. Clearly ϕ is onto; because for $(m + \boldsymbol{M}1, m + \boldsymbol{M}2) \in \boldsymbol{M/M_1} \otimes \boldsymbol{M/M_2}$ there exists $m \in \boldsymbol{M}$ such that $(m)\, \phi = (m + \boldsymbol{M}1, m + \boldsymbol{M}2)$. Thus by first isomorphism theorem we have $\boldsymbol{M/ker\phi} \cong \boldsymbol{M/M_1} \otimes \boldsymbol{M/M_2}$.

Now $\quad \boldsymbol{ker}\phi = \{m \in \boldsymbol{M} \mid (m)\, \phi = (\boldsymbol{M}1, \boldsymbol{M}2)\}$

$= \{m \in \boldsymbol{M} \mid (m + \boldsymbol{M}1, m + \boldsymbol{M}2) = (\boldsymbol{M}1, \boldsymbol{M}2)\}$

$= \{m \in \boldsymbol{M} \mid m + \boldsymbol{M}1 = \boldsymbol{M}1, m + \boldsymbol{M}2 = \boldsymbol{M}2\}$

$= \{m \in \boldsymbol{M} \mid m \in \boldsymbol{M}1 \text{ and } m \in \boldsymbol{M}2\}$

$= \{m \in \boldsymbol{M} \mid m \in \boldsymbol{M}1 \cap \boldsymbol{M}2\}$

$= \boldsymbol{M}1 \cap \boldsymbol{M}2$.

Hence $\boldsymbol{M/M_1} \cap \boldsymbol{M_2} \cong \boldsymbol{M/M_1} \otimes \boldsymbol{M/M_2}$.

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