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Uncoupled lateral buckling energy functional for beams with dual symmetry

***P.N. Jiki,**

Civil Engineering Department, University of Agriculture Makurdi, Nigeria.

****Corresponding author:*** *Dr. P.N. Jiki, Civil Engineering Department, University of Agriculture Makurdi, Nigeria. Email: peterjiki@yahoo.com*

Abstract

This paper is concerned with uncoupling the lateral buckling energy functional for thin walled beams of dual symmetric sections so that only one solution function would be used in the solution process without appreciable loss of accuracy. This is achieved by consideration of equilibrium of forces in the lateral buckling plane as well as equilibrium in the vertical plane, for the uncoupled equilibrium state. Examples are presented to show the validity of the proposed functional. Results of analysis using the proposed functional are compared with exact values in the literature and are within two percent (2 %) of the exact values. It is concluded that the proposed functional is suitable for use in the calculation of lateral buckling strengths for beams with dual sections.

Keywords: Energy functional, Buckling, Lateral, Beams, uncoupled.

Notations:

E- Young's Modulus, G - Shear modulus, I Total potential energy functional, I_x , I_y- Second moments of area about x and y axis, I_w- Warping constant, I- Torsion constant, L- Beam length, M- Applied moment, M_x- Bending moment distribution about x axis, M_y- Bending moment distribution about y axis, M_{cr} - Critical buckling moment, M_T- Torsional moment, P- Applied transverse load, uDisplacement in the horizon (x, z) plane, V-Displacement in the vertical (y, z) plane, X,Y,Z Coordinate axes, β-Angle of twist

Introduction

The usual procedure in solution of problems in structural mechanics is to formulate the differential equation in terms of displacement or response function. The solution obtained is exact if exact response function is known. However, in practice, it is difficult to obtain the exact solution for most of the problems encountered. An approximate or a trial solution function is used instead of the exact solution function and the result obtained is upper bound on the exact solution, depending however on the accuracy of the problem formulation (Hsu & Wu, 1998).

An alternative to formulations using classical differential equation methods is to device energy functional in terms of displacement or response functions (Akhaee *et al*., 2009). Like the classical differential equation methods, the exact solution functions in this case are also difficult to obtain the trial solution functions are usually substituted into the energy functional and a minimisation process such as the Rayleigh-Ritz process is carried out to furnish equilibrium equations which are solved to obtain the parameters of interest. In carrying out lateral buckling stability analysis of beams, the above problem of finding exact solution functions and in addition, coupling of displacements are experienced. There is therefore a need to express the proposed functional in terms of one solution function if possible.

The problem of lateral buckling of I-section beams was examined by Hartz (1965) in which discrete elements were used to formulate and solve the coupled lateral buckling equations. Attard (1990) has developed a general non-dimensional equation for lateral buckling in which a general thin-walled section was considered but no attention was paid to the uncoupling of the coupled energy functional. Recently, Jiki (2007) has studied the behaviour of pre-cracked beam-columns using Liapunov's second method but did not uncouple the dual symmetric beams and beam-columns analyzed. Using the energy functional Attard (1986) employed the finite element method to solve the non-linear lateral buckling (Demanet & Ying,

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2007). The non-linear effect considered in this case was initial curvature but not geometric effect, etc. Furthermore, there is limited literature on non-linear lateral buckling generally and none has been found on the solution of uncoupled lateral buckling problem in particular (Do & Vetterli, 2005).

The purpose of the present paper is to show that the lateral buckling energy functional can be expressed in terms of only one displacement or response function. This will be very useful for design applications as the critical moment obtained using the uncoupled functional is slightly less than that obtained for the coupled functional. But this is on the safe side as far as design is concerned. It will also be used for finite element analysis and computer simulation of large systems. It will also ease the problem of looking for exact solution functions as only one would be needed for the proposed uncoupled functional. The resulting energy functional is also nonlinear.

Fundamental differential equations

An I-beam, representing a dual symmetric beam with a general loading, coordinate axes and a force system is shown in Fig. 1. The x and y coordinates coincide with the principal coordinates of the beam section are also shown in figure 1 and the z-coordinate coincides with the longitudinal axis of the beam (Jayalakshmi *et al.*, 2001). After buckling, the beam undergoes a vertical displacement *v*, a lateral displacement *u* and a twist about the *z* axis whose angle is given by β. The beam can also warp longitudinally. At buckling, the differential equations of equilibrium are (Winter, 1943, Timoshenko and Gere, 1961):

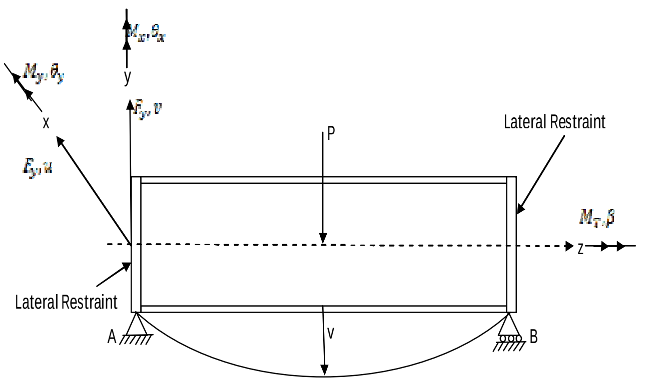

Fig. 1: Shows vertical bending of an I-beam with end restraints against lateral buckling. This bending represents the first equilibrium state of beam.

$$EI_x \frac{d^2y}{dx^2} - M_x = 0$$

$$EI_y \frac{d^2u}{dz^2} - M_x\beta = 0$$

$$GJ\frac{d^2\beta}{dz} - EI_w \frac{d^3\beta}{dz^3} - M_x \frac{du}{dz} = 0$$

Equation (1) shows that bending in the vertical plane, i.e. y axis is uncoupled with the other deformation modes and hence can be solved separately. This means that for doubly symmetric sections, two equilibrium states exist. The first equilibrium state exists in the vertical bending and the second state exists in the lateral buckling. This simplifies the analysis for lateral buckling of beams of symmetric sections as any loading combination can be applied and once the moment distribution on the beam can be found, it would be used as a load or disturbance for lateral buckling analysis using equations (2) and (3) respectively. See also Fig. 2 and 3 below.

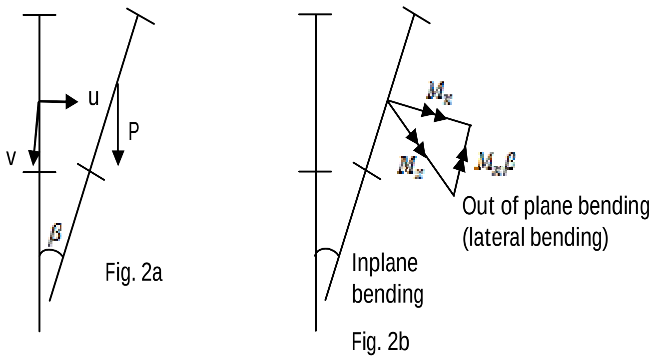

Fig. 2a: Shows section of the beam at the second equilibrium state. The beam bends laterally as well as twists. Fig. 2b: Shows contribution of vertical bending

Existing energy functional for lateral buckling of dual symmetric Sections

A thin-walled beam of a dual symmetric section that has undergone lateral buckling may be subjected to the following forms of energies (Timoshenko and Gere, 1961).

(a) Bending energy due to bending in the vertical plane and potential energy due to applied loads.

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(b) Lateral bending energy.

(c) Shear deformation energy due to twist of the beam.

(d) Warping deformation energy due to lateral flexural bending and twisting of the beam. The warping deformation energy however, depends on the end fixity condition of the beam.

The main contribution and of the energy functional is the bending moment distribution and of course the position of the applied force i.e. whether it is on top flange or on the bottom flange. It is possible to apply loads through the bottom flange of the beam. It is assumed in the present article that the load is applied to the neutral axis of the beam.

The total potential energy functional for a dual symmetric beam section in this case is given as:

$$\pi_1 = \frac{1}{2}\int EI_x(v'')^2 dz + \frac{1}{2}\int EI_y(u'')^2 dz + \frac{1}{2}\int GJ(\beta')^2 dz$$

$$+\frac{1}{2}\int EI_w(\beta'')^2 dz - \int M_x\beta u'' dz \qquad (4)$$

in which primes denote partial differentiation with respect to z. Equation (4) is coupled in β and u. In practice, the exact form of β and u may be difficult to obtain; so the fewer the solution functions are, the easier or the better for the analyst

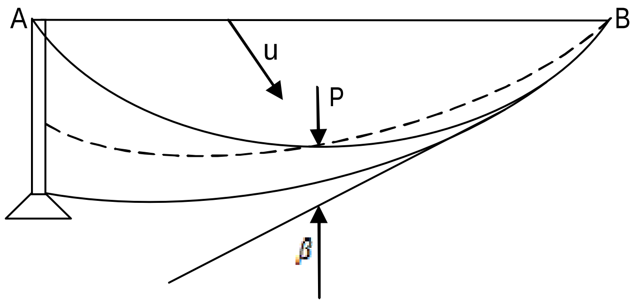

Fig. 3: Shows the beam at the second equilibrium state. The beam bends laterally as well as twists. The fibres at the middle of the beam may also warp.

as he has to worry for lesser solution functions. In the next section, the uncoupled form of equation (4) will be derived.

Proposed uncoupled lateral buckling energy functional for beams of dual symmetric Sections

Starting with equation (1) and following the argument that equation (1) is uncoupled and does not affect lateral buckling directly but through the bending moment distribution, the first term of equation (4) can be dropped to get:

$$\pi_2 = \frac{1}{2}\int EI_y(u'')^2 dz + \frac{1}{2}\int GJ(\beta')^2 dz$$

$$+\frac{1}{2}\int EI_w(\beta'')^2 dz - \int M_x\beta u'' dz \qquad (5)$$

from equation (2) the variable u'' can be obtained as:

$$\frac{d^2u}{dz^2} = \frac{M_x\beta}{EI_y} \qquad (6)$$

Substitution of equation (6) into equation (5) gives the energy functional uncoupled as:

$$\pi_2 = \frac{1}{2}\int EI_y\left(\frac{M_x}{EI_y}\right)^2 dz + \frac{1}{2}\int GJ(\beta')^2 dz$$

$$+\frac{1}{2}\int EI_w(\beta'')^2 dz - \int M_x\beta\left(\frac{M_x}{EI_y}\right) dz \qquad (7)$$

$$= \frac{1}{2}\int GJ(\beta')^2 dz + \frac{1}{2}\int EI_w(\beta'')^2 dz - \frac{1}{2}\int \frac{M_x^2\beta^2}{EI_y} dz \qquad (8)$$

Equation (8) is the required lateral buckling energy functional with one variable viz β. It must be emphasized here that the only advantage gained in using equation (8) instead of equation (5) is the ease of solution of equation (8) by using only β as a solution function.

Minimization of the total potential energy functional by the Rayleigh-Ritz Method

It has been said in the introductory section that substitution of an approximate solution function i.e. a trial solution function, into the total

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potential energy functional results in an approximate analysis in addition to that, it will be noticed that if a generalised trial solution function is used the result obtained will describe generalised conditions on the solution domain. The unknowns in this case are the generalised constants coefficients of the solution function. This is one of the differences between classical Rayleigh-Ritz and finite element method of analysis. In finite element analysis only nodal quantities are considered as the unknowns in the solution process and hence such quantities are no longer generalised.

For the beam problem considered herein, the variation of the total potential energy functional of either equations (5) or (8) is taken after which trial solution functions or function is substituted into the functional of either equation (5) or equation (8). Rayleigh-Ritz minimization condition is applied to obtain the parameter of interest. In the present case, the critical buckling moment is obtained. However, care is usually taken to select a solution function that satisfies both the geometric and essential boundary conditions for the problem.

Applications

Example 1:

In this example we study the lateral buckling behaviour of a simply supported beam loaded with a constant moment M. The beam has a rectangular section; and that warping is prevented. Also the following displacement functions are assumed: (Timoshenko and Gere, 1961):

$$u = a(z^2 - Lz) \quad u'' = 2a \tag{9}$$

$$\beta = b(z^2 - Lz) \quad \beta' = b(2z - l) \tag{10}$$

In which a and b are displacement amplitudes. Substitution of equation (9) and (10) into the coupled functional of equation (5) and carrying out the Rayleigh-Ritz minimisation process, the critical moment is obtained as:

$$M_{cr} = \frac{3.46}{L}\left(EI_y GJ\right)^{1/2} \qquad (11$$

This is 31.84% higher than exact value of $\frac{\pi}{2}\left(EI_y GJ\right)^{1/2}$ for the same problem presented by Timoshenko and Gere [5].

Appendix contains details of the Rayleigh-Ritz process.

Example 2:

In this example the sample problem of example 1 is solved using the proposed uncoupled functional of equation (8) and with one displacement function β of equation (10). Therefore substitution of equation (10) into equation (8) and carrying out the Rayleigh-Ritz minimisation process gives the critical moment M_{cr} as:

$$M_{cr} = \frac{3.16}{L}\left(EI_y GJ\right)^{1/2} \tag{12}$$

This is only 1.84% higher that the exact value of $\frac{\pi}{2}\left(EI_y GJ\right)^{1/2}$ (Timoshenko and Gere, 1961).

Comparing equation (11) and equation (12) it is seen that equation (12) which is the result from the proposed uncoupled functional is 8.67% less than that of the coupled functional given in equation (11). Again details of the Rayleigh-Ritz minimisation process is given in example (2) of appendix 1.

Conclusion

In this paper it has been shown that the coupled lateral buckling energy functional can be uncoupled without adversely reducing the accuracy of the solution for lateral buckling of dual symmetric beams. The view point was confirmed when a simply supported beam loaded with a constant moment was analysed using the existing coupled functional and the proposed uncoupled functional. It was found, however, that the two results differ by about 8.67% and that the result of the proposed functional was on the safe side as far as design was concerned, that is, it was lower than that obtained for the coupled functional.

Thus the proposed functional is therefore acceptable as results using the functional are closer to the exact values and are on the safe side as far as

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design is concerned. We therefore recommend the use of the functional proposed herein for the calculation of the lateral buckling strengths of dual symmetric beams.

Appendix

Example 1.

Consider a simply supported beam loaded with a constant movement M. The beam considered in the present investigation is fabricated from a thin-walled rectangular section and that warping is prevented. The following displacement functions are taken from (Timoshenko and Gere, 1961).

$$u = a(z^2 - Lz) \quad u'' = 2a \qquad (A1.1)$$

$$\beta = b(z^2 - Lz) \quad \beta' = b(2z - l) \qquad (A1.2)$$

in which the primes represent differentiation with respect to z and L is the length of the beam.

In variational notation, the functional of equation (5) becomes:

$$\delta\pi_1 = \int_0^1 EI_y u'' \delta u'' dz + \int_0^1 GJ\beta' \delta\beta' dz - \int_0^1 M_x \delta(\beta u'') dz \qquad (A1.3)$$

$$= \delta I_1 - \delta I_2 \qquad (A1.4)$$

in which

$$\delta I_1 = \int_0^1 EI_y u'' \delta u'' dz + \int_0^1 GJ\beta' \delta\beta' dz \qquad (A1.5)$$

$$\delta I_2 = \int_0^1 M_x \delta(\beta u'') dz \qquad (A1.6)$$

Substitution of u'' and β' into equation (A1.3) and differentiating partially with respect to a and b we have:

$$\frac{\delta I_1}{\delta a} = EI_y \int_0^1 (2a)(2)dz + 0 = 4EI_y aL \qquad (A1.7)$$

$$\frac{\delta I_1}{\delta b} = 0 + GJ \int_0^1 b(2z - L)^2 dz = \frac{1}{3} GJ\, bL^3 \qquad (A1.8)$$

$$\frac{\delta I_2}{\delta a} = \int_0^1 Mb(z^2 - Lz)(z)dz = -\frac{MaL^3}{3} \qquad (A1.9)$$

$$\frac{\delta I_2}{\delta b} = \int_0^1 Mb(z^2 - Lz)(z)dz = -\frac{MaL^3}{3} \qquad (A1.10)$$

Equations (A1.7) to (A1.10) are written in matrix form as:

$$\begin{bmatrix} 4EI_y L & \frac{-ML^3}{3} \\ \frac{-ML^3}{3} & \frac{GJL^3}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \qquad (A1.11)$$

For a non trivial solution to exist, the determinant of the equation (A1.11) should vanish, i.e.

$$\frac{4}{3} EI_y GJL^4 - \frac{M^2 L^6}{9} = 0 \qquad (A1.12)$$

Solving for M^2 gives:

$$M^2 = \frac{(12EI_y GJ)}{L^2} \qquad (A1.13)$$

or

$$M_{cr} = \frac{3.46}{L} (EI_y . GJ)^{1/2} \qquad (A1.14)$$

This is 31.84% higher than exact value of $\frac{\pi}{2}(EI_y GJ)^{1/2}$ for the same problem presented by Timoshenko and Gere 1961).

Example 2

For the purpose of comparison,the same problem of example (1) above is solved using the

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proposed uncoupled energy functional of equation (8). By taking variations of equation (8), the uncoupled total potential energy functional for lateral buckling being considered becomes:

$$\delta \pi_2 = \int_0^1 GJ\beta' \delta\beta' dz - \int_0^1 \frac{M_x^2 \beta \delta\beta}{EI_y} dz \qquad (A1.15)$$

$$= I_3 - I_4 \qquad (A1.16)$$

Substitution of the function β or its appropriate derivatives from equation (A1.2) and differentiating partially with respect to b gives:

$$\frac{\delta I_3}{\delta b} = GJ \int_0^L b(2z-L)^2 dz = \frac{GJbL^3}{3} \qquad (A1.17)$$

$$\frac{\delta I_4}{\delta b} = \frac{M^2 bL^5}{30EI_y} \qquad (A1.18)$$

The equilibrium equation is:

$$\frac{GJbL^3}{3} - \frac{M^2 bL^5}{30EI_y} = 0 \qquad (A1.19)$$

or

$$M_{cr} = \frac{3.16}{L}(GJ\, EI_y)^{1/2} \qquad (A1.20)$$

This is only 1.84% higher that the exact value of $\frac{\pi}{2}(EI_y GJ)^{1/2}$ (Timoshenko and Gere, 1961). The result of equation (A1.14) is obtained in example (1) for the coupled functional. The proposed functional is therefore worth testing further for design applications.

References

1. Akhaee MA, Sahraeian SME, Sankur B & Marvasti F (2009) Robust scaling-based image water marking using maximum-likelihood decoder with optimum strength factor. *IEEE Trans. Multimedia*. 11(5), 822–833.
2. Demanet L & Ying L (2007) Wave atoms and sparsity of oscillatory patterns. *Appl. Comput. Harmon. Anal*, 23, 368.
3. Do MN & Vetterli M (2005) The contour let transform: An efficient directional multi resolution image representation. *IEEE Trans. Image Process.* 14(12), 2091–2106.
4. Hsu CT & Wu JL (1998) Multiresolution water marking for digital images. *IEEE Trans. Circuit Syst*. 45(8) 1097–1101.
5. Jayalakshmi M, Merchant SN, & Desai UB (2006) Digital watermarking in contour let domain. *In Proc. 18th Int. Conf. Pattern Recognition*. 3, 861–864.
6. Hartz, BJ (1965) Matrix formulation of structural stability problems. J. Struct. Div. ASCE. 91(ST6): 141-157.
7. Attard, MM (1986) Lateral buckling analysis of beams by FEM. Computers and Structures, 21: 286-296.
8. Timoshenko, SP & Gere, JM (1961) Theory of Elastic Stability, McGraw-Hill, Inc., New York, NY.
9. Jiki, PN (2007). Buckling analysis of pre-cracked beam-columns by Liapunov's second method. *European J. Mechanics*. A /Solids. 26: 503-518.