Reason- A Technical Journal
ISSN 2277-1654
Volume XX 2021

UNIT VECTOR RELATIONS VIA DIRECTION COSINES

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Paper submitted on November 27 2021, Accepted after revision on March 11 2022
DOI:10.21843/reas/2021/57-61/212374

Abstract: Conversion of a vector from a coordinate system to another is done via the dot product operation. Unit vector relationships have been either studied by the vector projection method or by a set of complex geometric relationship. Both of these conventional methods are rather lengthy & time-consuming and are moreover difficult to recall. In this paper, through a step by step approach employing direction cosines, the authors were able to find the unit vector conversions between the rectangular and the spherical system efficiently. A densely labelled graph showing all variable relations is required from which the results precipitate coincidentally.

Keywords: Cartesian, coordinate system, vector conversion, dot product, electromagnetism, rectangular, spherical, direction cosines, unit vector.

1. INTRODUCTION

Vector algebra is a crucial mathematics domain. It is heavily employed in the study and presentation of electric and magnetic phenomena. Electromagnetic problems can be made easier to tackle by addressing them in a suitable choice of coordinate system.

In particular, a unit vector conversion between a spherical system and the rectangular coordinate system [1] is done using a set of complex geometric figures. This method causes delays, owing to the comprehension time involved, relational longevities and complexity. Projection method [2] is an alternate approach to perform this conversion. Unfortunately, this method too requires prior knowledge of vector relations. And there has always been a need to simplify this overwhelming relational theory.

In this paper, an alternative, but an efficient approach, is presented: The Method. The authors begin with putting together the basic equations and variables of the three coordinate systems (Cartesian/ Rectangular, Cylindrical and Spherical) and device a way to relate them on a common platform. In this process, the authors have successfully reached in implementing a relatively efficient as well as an easier way to compute the spherical unit vectors in terms of the Cartesian unit vectors.

Before diving into the explanation and application of the method, it is worth furnishing a foreground theory to accommodate a wide range of readers to benefit from this work. A summary of the standard vector relation theory is made available in Table 1 and Table 2 for a ready reference.

Basically, a vector component can be expressed in a different system by taking the dot product of a unit vector of the query component and a known destination vector as shown in the Table 2(a). Then, the vector equations of the Table 2(a), can be collectively presented in matrix forms as shown in Table 2(b).

Direction cosines are the cosines of angles a vector makes with the coordinate axis. A brief review of direction cosines (cos α, cos γ and cos β) is presented in Table 2(c).

Having a brief knowledge of the electromagnetic phenomena [3-9], presented through vectors, by sharing both the magnitude and the pathways information as measurable physical quantities; it is easy to comprehend the information tabulated in the initial tables listed in this paper. So, the authors are not repeating the explanations for the Table 1(a-c) and the Table 2(a-c) which are an integral part of the primary/ standard references [1] and [2].

Having listed sufficient underlying fundamentals, the authors can move onto building this theory for finding the dot products of unit vectors in the rectangular and the spherical system in the next section.

Table I	Coordinate System variable relations
I-A)	**Cylindrical and Rectangular coordinate system**
$\rho = \sqrt{x^2+y^2}$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$ $z = z$	$x = \rho\cos(\phi)$ $y = \rho\sin(\phi)$ $z = z$
I-B)	**Spherical and Rectangular coordinate system**
$r = \sqrt{x^2+y^2+z^2}$ $\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$	$x = r\sin(\theta)\cos(\phi)$ $y = r\sin(\theta)\sin(\phi)$ $z = r\cos(\theta)$
I-C)	**Cylindrical and Spherical coordinate system**
$r = \sqrt{\rho^2+z^2}$ $\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$ $\phi = \phi$	$\rho = r\sin(\theta)$ $z = r\cos\theta$ $\phi = \phi$
where, (x, y, z) are rectangular coordinates (ρ, ϕ, z) are cylindrical coordinates (r, θ, ϕ) are spherical coordinates	

Table II-A	Vector Components for different systems
$A_\rho = a_\rho \cdot (A_x a_x + A_y a_y + A_z a_z) =$ $A_x a_x \cdot a_\rho + A_y a_y \cdot a_\rho + A_z a_z \cdot a_\rho$	
$A_\phi = a_\phi \cdot (A_x a_x + A_y a_y + A_z a_z) =$ $A_x a_x \cdot a_\phi + A_y a_y \cdot a_\phi + A_z a_z \cdot a_\phi$	
$A_z = a_z \cdot (A_x a_x + A_y a_y + A_z a_z) =$ $A_x a_z \cdot a_x + A_y a_z \cdot a_y + A_z a_z \cdot a_z$	
$A_r = a_r \cdot (A_x a_x + A_y a_y + A_z a_z) =$ $A_x a_x \cdot a_r + A_y a_y \cdot a_r + A_z a_z \cdot a_r$	
$A_\theta = a_\theta \cdot (A_x a_x + A_y a_y + A_z a_z) =$ $A_x a_x \cdot a_\theta + A_y a_y \cdot a_\theta + A_z a_z \cdot a_\theta$	
$A_x, A_y, A_z, A_\rho, A_\phi, A_r$ & A_θ are vector components and $a_x, a_y, a_z, a_r, a_\phi$ and a_θ are the unit vectors.	

Table II-B	Vector Components in matrix form
$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} a_x \cdot a_\rho + a_y \cdot a_\rho + a_z \cdot a_\rho \\ a_x \cdot a_\phi + a_y \cdot a_\phi + a_z \cdot a_\phi \\ a_x \cdot a_z + a_y \cdot a_z + a_z \cdot a_z \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ $\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} a_x \cdot a_r + a_y \cdot a_r + a_z \cdot a_r \\ a_x \cdot a_\theta + a_y \cdot a_\theta + a_z \cdot a_\theta \\ a_x \cdot a_\phi + a_y \cdot a_\phi + a_z \cdot a_\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$	

Table II-C	Direction Cosines (l, m and n) defined for an arbitrary vector along the component A_r with (x,y,z) coordinates as shown in Figure I.
$l = \cos\alpha = \frac{x}{Hx^2+y^2+z^2}$ $m = \cos\gamma = \frac{y}{Hx^2+y^2+z^2}$ $n = \cos\beta = \frac{z}{Hx^2+y^2+z^2}$	

2. THE METHOD INTRODUCED

A sketch of an amply labeled coordinate system is prepared and attached as Fig. 1. Notice that all the angles α, γ and β directions are labelled in this one single diagram. All direction cosines are defined on an arbitrary vector along the component A_r that has angles a, g and b with the coordinate arms x, y and z respectively. The unit vector dot products presented in the second row of the Table 2(b) can now be completely revealed using Fig. 1. As such, this method aims at answering the results of the following dot products:

$$\hat{x}\cdot\hat{r}\ ;\ \hat{y}\cdot\hat{r}\ ;\ \hat{z}\cdot\hat{r}$$
$$\hat{x}\cdot\hat{\theta}\ ;\ \hat{y}\cdot\hat{\theta}\ ;\ \hat{z}\cdot\hat{\theta}$$
$$\hat{x}\cdot\hat{\phi}\ ;\ \hat{y}\cdot\hat{\phi}\ ;\ \hat{z}\cdot\hat{\phi}$$

The concept of direction cosines can be extended to other axis also, such as δ being an angle between a unit vector along the x coordinate arm and a unit vector along the $\hat{\theta}$ coordinate as shown in Fig. 2(b). Note that θ is an elevation angle of the spherical system which is subtended by the head of the conventional radial vector, projected on the z axis of the rectangular coordinate system [2]. Similarly, σ is an angle between a unit vector y and a unit vector along the $\hat{\theta}$ coordinate. The rectangular (x, y & z), cylindrical (ρ, ϕ & z)) and the spherical coordinates (r, θ & ϕ) are all experienced together in Fig. 1. For the derivation of unit vector dot products, between the rectangular and the spherical coordinate systems, the authors simply employ direction cosines. Further, the authors introduce the insufficiencies as multiples to bring the results back to a trigonometric ratio in the other corresponding system.

For instance, in computing the first listed dot product on Table 3, the authors have $\hat{x}\cdot\hat{r} = |\hat{x}|\,|\hat{r}|\cos(\alpha) = \cos(\alpha)$. Here one needs to realize the equivalent spherical coordinate component. With the exhaustive labelling in Fig. 1, one can easily infer that $\cos(\alpha) = \left(\frac{X}{R}\right)$ where X is the arm length along x axis, the arm obtained by dropping a perpendicular on x-axis from the head of a radial vector running along $\hat{r}$ direction and R being the magnitue of the radial vector itself. Now, one can multiply $\left(\frac{X}{R}\right)$ with one, thus one essentially does not change the equation. And 1 can be put equal to $\left(\frac{\rho_c}{\rho_c}\right)$, where ρ_c can be an arbitary variable. So, the authors have, $\left(\frac{X}{R}\right)(1) = \left(\frac{X}{R}\right)\left(\frac{\rho_c}{\rho_c}\right)$. Rearranging gives, $\left(\frac{X}{R}\right)\left(\frac{\rho_c}{\rho_c}\right) = \left(\frac{X}{\rho_c}\right)\left(\frac{\rho_c}{R}\right)$. Now again from Fig. 1, notice that $\cos(\phi) = \left(\frac{X}{\rho_c}\right)$ and $\sin(\theta) = \left(\frac{\rho_c}{R}\right)$ where ρ_c is the cylindrical radius in the xy plane extending diagonally between the x and y arms. Thus the authors end up finding, $\hat{x}\cdot\hat{r} = \cos(\phi)\sin(\theta)$. Similarly, one may deduce all others. All such conversions in the work are compiled and listed in the Table 3 and are self explanatory in nature.

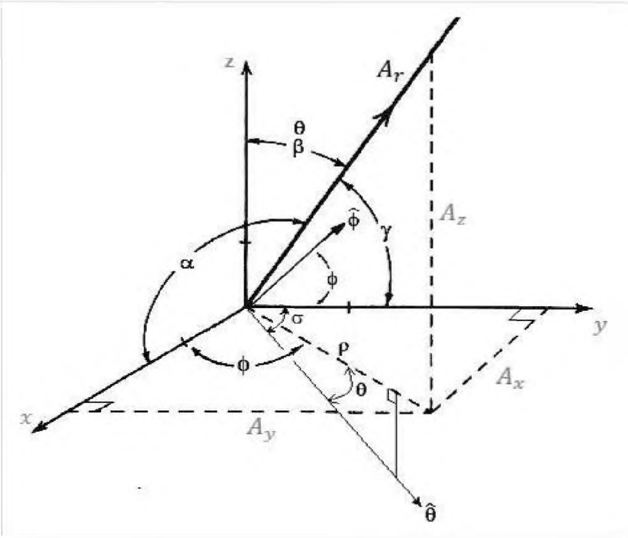

Fig. 1. Coordinate axis and their angles with each other

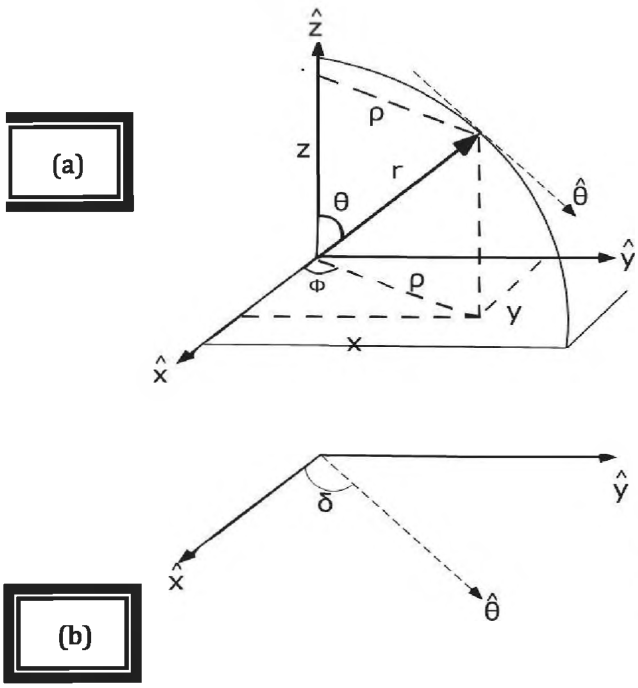

Fig. 2. Concept of direction cosines extended to different axes

3. RESULTS

Table 3 : Dot products using direction cosines

$\hat{x} \cdot \hat{r} = \cos(\alpha) = \left(\frac{X}{R}\right)\left(\frac{\rho_c}{\rho_c}\right) = \cos(\phi)\sin(\theta)$
$\hat{x} \cdot \hat{\theta} = \cos(\delta) = \left(\frac{X}{\theta_c}\right)\left(\frac{\rho_c}{\rho_c}\right) = \cos(\phi)\cos(\theta)$
$\hat{x} \cdot \hat{\phi} = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin(\phi)$
$\hat{y} \cdot \hat{r} = \cos\beta = \left(\frac{Y_c}{R}\right)\left(\frac{\rho_c}{\rho_c}\right) = \cos\left(\frac{\pi}{2} - \phi\right)\cos\left(\frac{\pi}{2} - \theta\right)$ $= \sin(\phi)\sin(\theta)$
$\hat{y} \cdot \hat{\theta} = \cos(\sigma) = \left(\frac{Y}{\theta_c}\right)\left(\frac{\rho_c}{\rho_c}\right) = \sin(\phi)\cos(\theta)$
$\hat{y} \cdot \hat{\phi} = \cos(\phi)$
$\hat{z} \cdot \hat{r} = \cos(\theta)$
$\hat{z} \cdot \hat{\theta} = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin(\theta)$
$\hat{z} \cdot \hat{\phi} = \cos\left(\frac{\pi}{2}\right) = 0$
Note: X, Y, θ_c, ρ_c and R are components/variables.

Having generated Table 3 using vector math and trigonometric relations, one is now in a position to define the unit vector relations between the basic unit vectors, $\hat{X}, \hat{Y}, \hat{Z}$ AND $\hat{R}, \hat{\Theta}, \hat{\Phi}$ as follows [2]:

$$\hat{x} = \cos(\varphi)\sin(\theta)\,\hat{r} + \cos(\varphi)\cos(\theta)\,\hat{\theta} - \sin(\varphi)\,\hat{\varphi}$$
$$\hat{y} = \sin(\varphi)\sin(\theta)\,\hat{r} + \sin(\varphi)\cos(\theta)\,\hat{\theta} + \cos(\varphi)\,\hat{\varphi}$$
$$\hat{z} = \cos(\theta)\hat{r} - \sin(\theta)\,\hat{\theta}$$

These results can be collected from Table 3. It is evident that this method is far simpler and efficient in computing the dot products with the help of just one Fig. 1. The diagram shows all necessary details of the three coordinate systems discussed in this paper. Also it is obvious that in order to compute the unit vector conversions using the graphical tool, a human subject does far better than a machine implementation due to the limited cognition and artificial intelligence capability of today's machines. One has also to encourage oneself and the readers may develop software implementations of this work and evaluate its applicability in automation.

4. CONCLUSION

It is possible to break down the dot products of two unit vectors into component ratios by using direction cosines. This method is far simpler and time efficient than the methods in practice, and thus simplifies the vector algebra that is useful in expressing electromagnetic theory.

REFERENCES

[1] Hayt, W.H., Buck, J.A. and Akhtar, Engineering Electromagnetics, 8th Ed., Tata McGraw-Hill.

[2] Sadiku, M., Elements of Electro-magnetics, 6th Ed., Oxford University Press.

[3] Serway, R. A., Physics for Scientists & Engineers with Modern Physics, 4th Ed., Saunders College Pub, Philadel-phia, USA.

[4] Stroud, K.A. and Booth, D., Engineering Mathematics, 7th Ed., Red Globe Press.

[5] A collection of materials for physics students and instructors, accessed in November 2021. https://www.cpp.edu/~ajm/materials/delsph.pdf

[6] Snyder, J.P., Map Projections: A Working Manual, Geological Survey (U.S.), Report Number 1395, p.37, 1987.

[7] Gauss, K.F., General Investigations of Curved Surfaces of 1827 and 1825, The Princeton University Library, Translated in 1902.

[8] Zahn, M., Electromagnetic Field Theory, MIT Open Course Ware, http://ocw.mit.edu, accessed in January 2022.

[9] Zahn, M., Electromagnetic Field Theory: A Problem Solving Approach, Malabar, F.L.: Krieger Publishing Company, 2003.