The Number of Vector Partitions of \( n \) (Counted According to the weight) with the Crank \( m \)

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ABSTRACT
This article shows how to find all vector partitions of any positive integral values of \( n \), but only all vector partitions of 4, 5 and 6 are shown by algebraically. These must be satisfied by the definitions of crank of vector partitions.

PACs: 02.60.-X

Keywords: Vector partitions, Crank, Congruences, Modulo

INTRODUCTION
Here we discuss such a crank which in terms of a weighted count of what we call vector partitions. We give the definitions of \( \pi \), \#(\( \pi \)), \( \sigma(\pi) \), crank of vector partitions, weight of \( \tilde{\pi} \), \( N_v(m,n) \), \( N_v(m,t,n) \) and prove the partitions congruences moduli 5, 7 and 11 with the help of examples by finding all vector partitions of 4, 5 and 6, respectively. We analyze the generating functions for \( N_v(m,n) \) and \( N_v(m,t,n) \).

DEFINITIONS
\( \pi \) : A partition.
\#(\( \pi \)) : The number of parts of \( \pi \).
\( \sigma(\pi) \) : The sum of the parts of \( \pi \).

Crank of vector partitions: The number of parts of \( \pi_2 \) minus the number of parts of \( \pi_3 \).

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where \(\pi_2\) and \(\pi_3\) are unrestricted partitions in a vector partition \(\vec{\pi} = (\pi_1, \pi_2, \pi_3)\) of \(n\), if the sum of \(\vec{\pi}\) is \(s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3) = n\).

Weight of \(\vec{\pi}\): Weight of vector partition \(\vec{\pi}\) is defined as: \(\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}\).

\(N_v(m,n)\): The number of vector partitions of \(n\) (counted according to the weight \(\omega\)) with the crank \(m\).

\(N_v(m,t,n)\): The number of vector partitions of \(n\) (counted according to the weight \(\omega\)) with the crank congruent to \(m\) modulo \(t\).

**The Crank for Vector Partitions**

For a partition \(\pi\), let \(\#(\pi)\) be the number of parts of \(\pi\) and \(\sigma(\pi)\) be the sum of the parts of \(\pi\) with the convention \(\#(\phi) = \sigma(\phi) = 0\) for the empty partition \(\phi\) of 0 (Andrews, 1985), (Andrews and Garvan, 1988).

Let, \(\vec{V} = \{(\pi_1, \pi_2, \pi_3)|\pi_1\text{ is a partition into unequal parts } \pi_2, \pi_3\text{ are unrestricted partitions}\} \). We shall call the elements of \(\vec{V}\) vector partitions. For \(\vec{\pi} = (\pi_1, \pi_2, \pi_3)\) in \(\vec{V}\) we define the sum of parts, \(s\), a weight, \(\omega\), and a crank, \(c\), by;

\[
\begin{align*}
\omega(\vec{\pi}) &= (-1)^{\#(\pi_1)} \\
c(\vec{\pi}) &= \#(\pi_1) - \#(\pi_2)
\end{align*}
\]

We say \(\vec{\pi}\) is a vector partition of \(n\), if \(s(\vec{\pi}) = n\). For example, if \(\vec{\pi} = (1,1+1,1)\), then \(s(\vec{\pi}) = 4\), \(\omega(\vec{\pi}) = -1\), \(c(\vec{\pi}) = 1\) and \(\vec{\pi}\) is a vector partition of 4.

The number of vector partitions of \(n\) (counted according to the weight \(\omega\)) with the crank \(m\) is denoted by \(N_v(m,n)\) so that;

\[
N_v(m,n) = \sum \omega(\vec{\pi}) \text{ if } \vec{\pi} \in \vec{V}, \ s(\vec{\pi}) = n, \text{ and } c(\vec{\pi}) = m.
\]

We have 41 vector partitions of 4 are given in the following table:

<table>
<thead>
<tr>
<th>Vector partitions of 4</th>
<th>Weight (\omega(\vec{\pi}))</th>
<th>Crank (c(\vec{\pi}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\vec{\pi}_1 = (\phi, \phi, 4))</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>(\vec{\pi}_2 = (\phi, \phi, 3+1))</td>
<td>+1</td>
<td>-2</td>
</tr>
<tr>
<td>(\vec{\pi}_3 = (\phi, \phi, 2+2))</td>
<td>+1</td>
<td>-2</td>
</tr>
<tr>
<td>(\vec{\pi}_4 = (\phi, \phi, 2+1+1))</td>
<td>+1</td>
<td>-3</td>
</tr>
<tr>
<td>(\vec{\pi}_5 = (\phi, \phi, 1+1+1+1))</td>
<td>+1</td>
<td>-4</td>
</tr>
<tr>
<td>(\vec{\pi}_6 = (\phi, 1,3))</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_i$</td>
<td>$\phi_{i,j,k}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>-------</td>
<td>----------------</td>
<td>-------</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td>$\phi_{1,2,3}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>$\phi_{1,2,4}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_3$</td>
<td>$\phi_{1,2,5}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_4$</td>
<td>$\phi_{1,2,6}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_5$</td>
<td>$\phi_{1,2,7}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_6$</td>
<td>$\phi_{1,2,8}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_7$</td>
<td>$\phi_{1,2,9}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_8$</td>
<td>$\phi_{1,2,10}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_9$</td>
<td>$\phi_{1,2,11}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{10}$</td>
<td>$\phi_{1,2,12}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{11}$</td>
<td>$\phi_{1,2,13}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{12}$</td>
<td>$\phi_{1,2,14}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{13}$</td>
<td>$\phi_{1,2,15}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{14}$</td>
<td>$\phi_{1,2,16}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{15}$</td>
<td>$\phi_{1,2,17}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{16}$</td>
<td>$\phi_{1,2,18}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{17}$</td>
<td>$\phi_{1,2,19}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{18}$</td>
<td>$\phi_{1,2,20}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{19}$</td>
<td>$\phi_{1,2,21}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{20}$</td>
<td>$\phi_{1,2,22}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{21}$</td>
<td>$\phi_{1,2,23}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{22}$</td>
<td>$\phi_{1,2,24}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{23}$</td>
<td>$\phi_{1,2,25}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{24}$</td>
<td>$\phi_{1,2,26}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{25}$</td>
<td>$\phi_{1,2,27}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{26}$</td>
<td>$\phi_{1,2,28}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{27}$</td>
<td>$\phi_{1,2,29}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{28}$</td>
<td>$\phi_{1,2,30}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{29}$</td>
<td>$\phi_{1,2,31}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{30}$</td>
<td>$\phi_{1,2,32}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{31}$</td>
<td>$\phi_{1,2,33}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{32}$</td>
<td>$\phi_{1,2,34}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{33}$</td>
<td>$\phi_{1,2,35}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{34}$</td>
<td>$\phi_{1,2,36}$</td>
<td>+1</td>
</tr>
<tr>
<td>$\pi_{35}$</td>
<td>$\phi_{1,2,37}$</td>
<td>+1</td>
</tr>
</tbody>
</table>
From the above table we have,
\[ N_v(0,4) = \omega(\pi_9) + \omega(\pi_0) + \omega(\pi_{12}) + \omega(\pi_{13}) + \omega(\pi_{24}) + \omega(\pi_{26}) + \omega(\pi_{33}) + \omega(\pi_{40}) + \omega(\pi_{41}) \]
\[ = 1 + 1 + 1 + 1 - 1 - 1 - 1 + 1 = 1 \]  
(1)
The number of vector partitions of \( n \) (counted according to the weight \( \omega \)) with the crank congruent to \( k \) modulo \( t \) is denoted by \( N_v(k,t,n) \), so that;
\[ N_v(k,t,n) = \sum_{m=-\infty}^{\infty} N_v(m,t+k,n) = \sum \omega(\pi) ; \]  
(2)
if \( \pi \in \tilde{V} \), \( s(\pi) = n \), and \( c(\pi) \equiv k \pmod{t} \).

From the table we get;
\[ N_v(1,5,4) = \omega(\pi_5) + \omega(\pi_{11}) + \omega(\pi_{14}) + \omega(\pi_{16}) + \omega(\pi_{27}) + \omega(\pi_{28}) + \omega(\pi_{34}) + \omega(\pi_{38}) + \omega(\pi_{39}) \]
\[ = 1 + 1 + 1 + 1 - 1 - 1 - 1 + 1 + 1 = 1 \]  
(3)
By considering the transformation that interchanges \( \pi_2 \) and \( \pi_3 \) we have;
\[ N_v(m,n) = N_v(-m,n) . \]

We illustrate with an example;
\[ N_v(1,4) = \omega(\pi_{11}) + \omega(\pi_{14}) + \ldots + \omega(\pi_{39}) \]
\[ = 1 + 1 + 1 - 1 - 1 - 1 + 1 + 1 = 0 \]
and
\[ N_v(-1,4) = \omega(\pi_1) + \omega(\pi_7) + \ldots + \omega(\pi_{37}) \]
\[ = 1 + 1 + 1 - 1 - 1 - 1 + 1 + 1 = 0 \]
\[ \therefore \ N_v(1,4) = N_v(-1,4) . \]
Again,
\[ N_v(5 - 1,5,4) = N_v(4,5,4) = \omega(\pi_{20}) = 1 \]
\[ \therefore \ N_v(1,5,4) = N_v(5 - 1,5,4) \quad \text{by (3)}. \]
Generally we can write,
\[ N_v(m,t,n) = N_v(t - m, t, n) \]
The Generating Function for \( N_v(m,n) \)

The generating function for \( N_v(m,n) \) is:

\[
\prod_{n=1}^{\infty} \frac{(1-x^n)}{(1-z^n)(1-z^{-1}x^n)} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m,n) z^n x^n
\]

which was proved by Atkin and Swinnerton-Dyer (1954). By putting \( z = 1 \) in (4), we get:

\[
\prod_{n=1}^{\infty} \frac{(1-x^n)}{(1-x^n)(1-x^n)} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m,n) x^n
\]

\[\Rightarrow \sum_{n=0}^{\infty} P(n) x^n = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m,n) x^n\]

\[\therefore P(n) = \sum_{m=-\infty}^{\infty} N_v(m,n).\] (5)

Now we discuss it with an example;

R.H.S. = \( \sum_{m=-\infty}^{\infty} N_v(m,n) \)

\[= \sum_{m=-\infty}^{\infty} N_v(m,4)\]

\[= N_v(-4,4) + N_v(-3,4) + N_v(-2,4) + N_v(-1,4) + N_v(0,4) + N_v(1,4) + N_v(2,4)
\]

\[+ N_v(3,4) + N_v(4,4) + ...\]

\[= 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 + 1 = 5 = P(4) = \text{L.H.S.}\]

The Generating Function for \( N_v(0,n) \)

The generating function for \( N_v(0,n) \) is defined as:

\[
(1-x) \sum_{n=0}^{\infty} \frac{x^{n(n+2)}}{(x^2)_n}
\]

\[= (1-x) \left[ 1 + \frac{x^3}{(1-x)^2} + \frac{x^8}{(1-x)^2(1-x^2)^2} + \frac{x^{15}}{(1-x)^2(1-x^2)^2(1-x^3)^2} + ... \right]\]

\[= 1 - x + 0.x^2 + x^3 + x^4 + x^5 + x^6 + ...\]

\[= N_v(0,0) + N_v(0,1)x + N_v(0,2)x^2 + N_v(0,3)x^3 + N_v(0,4)x^4 + N_v(0,5)x^5 + N_v(0,6)x^6 + ...\]
\[
= \sum_{n=0}^{\infty} N_{V}(0,n)x^n.
\]

**Result**

\[N_{V}(k,5,5n+4) = \frac{P(5n+4)}{5}; \quad 0 \leq k \leq 4.\]

**Proof:** We prove the result with an example.

From the table 1 we get:

\[N_{V}(0,5,4) = \omega(\pi_1) + \omega(\pi_9) + \omega(\pi_{12}) + \omega(\pi_{13}) + \omega(\pi_{24}) + \omega(\pi_{26}) + \omega(\pi_{33}) + \omega(\pi_{40}) + \omega(\pi_{41}) = 1+1+1+1+1+1+1+1+1 = 1,
\]

\[N_{V}(1,5,4) = 1+1+1+1+1+1+1+1 = 1,
\]

\[N_{V}(2,5,4) = 1+1+1+1+1+1+1 = 1,
\]

\[N_{V}(3,5,4) = 1+1+1+1+1+1 = 1,
\]

\[N_{V}(4,5,4) = 1+1+1+1+1+1+1+1 = 1.
\]

\[
N_{V}(0,5,4) = N_{V}(1,5,4) = N_{V}(2,5,4) = N_{V}(3,5,4) = N_{V}(4,5,4) = 1 = \frac{P(4)}{5}, \text{ where } n = 0.
\]

In general we can write:

\[N_{V}(k,5,5n+4) = \frac{P(5n+4)}{5}; \quad 0 \leq k \leq 4.
\]

Hence the Theorem.

\[N_{V}(k,5,5n+4) = \frac{P(5n+4)}{5}; \quad 0 \leq k \leq 4.
\]

**Proof:** We prove the result with an example.

The vector partitions of 5 are given in the table below:

<table>
<thead>
<tr>
<th>Vector partitions of 5</th>
<th>Weight (\omega(\pi))</th>
<th>Crank ((\pi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi_1 = (\phi,\phi,5))</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>(\pi_2 = (\phi,\phi,4+1))</td>
<td>+1</td>
<td>-2</td>
</tr>
<tr>
<td>(\pi_3 = (\phi,\phi,3+2))</td>
<td>+1</td>
<td>-2</td>
</tr>
<tr>
<td>(\pi_4 = (\phi,\phi,3+1+1))</td>
<td>+1</td>
<td>-3</td>
</tr>
<tr>
<td>(\pi_5 = (\phi,\phi,2+2+1))</td>
<td>+1</td>
<td>-3</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|c|}
\hline
\pi_6 &= (\phi, \phi, 2 + 1 + 1 + 1) & +1 & -4 \\
\pi_7 &= (\phi, \phi, 1 + 1 + 1 + 1) & +1 & -5 \\
\pi_8 &= (5, \phi, \phi) & -1 & 0 \\
\pi_9 &= (\phi, 5, \phi) & +1 & 1 \\
\pi_{10} &= (\phi, 4 + 1, \phi) & +1 & 2 \\
\pi_{11} &= (4 + 1, \phi, \phi) & +1 & 0 \\
\pi_{12} &= (4, 1, \phi) & -1 & 1 \\
\pi_{13} &= (1, 4, \phi) & -1 & 1 \\
\pi_{14} &= (\phi, 4, 1) & +1 & 0 \\
\pi_{15} &= (\phi, 1, 4) & +1 & 0 \\
\pi_{16} &= (1, \phi, 4) & -1 & -1 \\
\pi_{17} &= (4, \phi, 1) & -1 & -1 \\
\pi_{18} &= (3 + 2, \phi, \phi) & +1 & 0 \\
\pi_{19} &= (\phi, 3 + 2, \phi) & +1 & 2 \\
\pi_{20} &= (3, 2, \phi) & -1 & 1 \\
\pi_{21} &= (2, 3, \phi) & -1 & 1 \\
\pi_{22} &= (\phi, 3, 2) & +1 & 0 \\
\pi_{23} &= (\phi, 2, 3) & +1 & 0 \\
\pi_{24} &= (3, \phi, 2) & -1 & -1 \\
\pi_{25} &= (2, \phi, 3) & -1 & -1 \\
\pi_{26} &= (\phi, 3 + 1 + 1, \phi) & +1 & 3 \\
\pi_{27} &= (3 + 1, 1, \phi) & +1 & 1 \\
\pi_{28} &= (1, 3 + 1, \phi) & -1 & 2 \\
\pi_{29} &= (\phi, 3 + 1, 1) & +1 & 1 \\
\pi_{30} &= (\phi, 1, 3 + 1) & +1 & -1 \\
\pi_{31} &= (3 + 1, \phi, 1) & +1 & -1 \\
\pi_{32} &= (1, \phi, 3 + 1) & -1 & -2 \\
\pi_{33} &= (3, 1 + 1, \phi) & -1 & 2 \\
\pi_{34} &= (\phi, 1 + 1, 3) & +1 & 1 \\
\hline
\end{array}
\]
\[
\begin{array}{|c|c|c|}
\hline
\tilde{\pi} & \text{Value} & \text{Value} \\
\hline
\tilde{\pi}_{35} = (\phi,3,1 + 1) & +1 & -1 \\
\tilde{\pi}_{36} = (3,\phi,1 + 1) & -1 & -2 \\
\tilde{\pi}_{37} = (\phi,2 + 2 + 1,\phi) & +1 & 3 \\
\tilde{\pi}_{38} = (1,2 + 2,\phi) & -1 & 2 \\
\tilde{\pi}_{39} = (\phi,2 + 2,1) & +1 & 1 \\
\tilde{\pi}_{40} = (\phi,1,2 + 2) & +1 & -1 \\
\tilde{\pi}_{41} = (1,\phi,2 + 2) & -1 & -2 \\
\tilde{\pi}_{42} = (2 + 1,2,\phi) & +1 & 1 \\
\tilde{\pi}_{43} = (2,2 + 1,\phi) & -1 & 2 \\
\tilde{\pi}_{44} = (\phi,2,2 + 1) & +1 & 1 \\
\tilde{\pi}_{45} = (\phi,2 + 1,2) & +1 & 1 \\
\tilde{\pi}_{46} = (2 + 1,\phi,2) & +1 & -1 \\
\tilde{\pi}_{47} = (2,\phi,2 + 1) & -1 & -2 \\
\tilde{\pi}_{48} = (\phi,2 + 2 + 1,\phi) & +1 & 4 \\
\tilde{\pi}_{49} = (\phi,2 + 1 + 1,1) & +1 & 2 \\
\tilde{\pi}_{50} = (\phi,1,2 + 1 + 1) & +1 & -2 \\
\tilde{\pi}_{51} = (1,2 + 1 + 1,\phi) & -1 & 3 \\
\tilde{\pi}_{52} = (1,\phi,2 + 1 + 1) & -1 & -3 \\
\tilde{\pi}_{53} = (2 + 1,1 + 1,\phi) & +1 & 2 \\
\tilde{\pi}_{54} = (\phi,2 + 1,1 + 1) & +1 & 0 \\
\tilde{\pi}_{55} = (\phi,1 + 1,2 + 1) & +1 & 0 \\
\tilde{\pi}_{56} = (2 + 1,\phi,1 + 1) & +1 & -2 \\
\tilde{\pi}_{57} = (\phi,1 + 1 + 1,2) & +1 & 2 \\
\tilde{\pi}_{58} = (\phi,2,1 + 1 + 1) & +1 & -2 \\
\tilde{\pi}_{59} = (2,1 + 1 + 1,\phi) & -1 & 3 \\
\tilde{\pi}_{60} = (2,\phi,1 + 1 + 1) & -1 & -3 \\
\tilde{\pi}_{61} = (\phi,1 + 1 + 1 + 1,\phi) & +1 & 5 \\
\tilde{\pi}_{62} = (\phi,1 + 1 + 1 + 1,1) & +1 & 3 \\
\tilde{\pi}_{63} = (\phi,1 + 1 + 1 + 1) & +1 & -3 \\
\hline
\end{array}
\]
\[
\begin{align*}
\vec{\pi}_{64} &= (1, \phi, 1 + 1 + 1 + 1) & -1 & -4 \\
\vec{\pi}_{65} &= (1, 1 + 1 + 1 + 1, \phi) & -1 & 4 \\
\vec{\pi}_{66} &= (\phi, 1 + 1 + 1 + 1) & +1 & -1 \\
\vec{\pi}_{67} &= (\phi, 1 + 1 + 1 + 1) & +1 & 1 \\
\vec{\pi}_{68} &= (1, 1 + 1 + 1) & -1 & -2 \\
\vec{\pi}_{69} &= (1, 1 + 1 + 1) & -1 & 2 \\
\vec{\pi}_{70} &= (1, 1 + 1 + 1) & -1 & 0 \\
\vec{\pi}_{71} &= (1, 1 + 1, 2) & -1 & 1 \\
\vec{\pi}_{72} &= (1, 2, 1 + 1) & -1 & -1 \\
\vec{\pi}_{73} &= (2, 1, 1 + 1) & -1 & 1 \\
\vec{\pi}_{74} &= (2, 1, 1 + 1) & -1 & -1 \\
\vec{\pi}_{75} &= (2, 2, 1) & -1 & 0 \\
\vec{\pi}_{76} &= (2, 1, 2) & -1 & 0 \\
\vec{\pi}_{77} &= (1, 2, 2) & -1 & 0 \\
\vec{\pi}_{78} &= (3, 1, 1) & -1 & 0 \\
\vec{\pi}_{79} &= (1, 3, 1) & -1 & 0 \\
\vec{\pi}_{80} &= (1, 1, 3) & -1 & 0 \\
\vec{\pi}_{81} &= (1 + 2, 1, 1) & +1 & 0 \\
\vec{\pi}_{82} &= (1, 1 + 2, 1) & -1 & 1 \\
\vec{\pi}_{83} &= (1, 1, 1 + 2) & -1 & -1
\end{align*}
\]

From this table we have;
\[
N_v(0, 7, 5) = \omega(\vec{\pi}_8) + \omega(\vec{\pi}_{11}) + \omega(\vec{\pi}_{14}) + \omega(\vec{\pi}_{15}) + \\
\omega(\vec{\pi}_{18}) + \omega(\vec{\pi}_{22}) + \omega(\vec{\pi}_{23}) + \omega(\vec{\pi}_{54}) + \omega(\vec{\pi}_{55}) + \\
\omega(\vec{\pi}_{70}) + \omega(\vec{\pi}_{75}) + \omega(\vec{\pi}_{76}) + \omega(\vec{\pi}_{78}) + \omega(\vec{\pi}_{79}) + \omega(\vec{\pi}_{79}) + \omega(\vec{\pi}_{80}) + \omega(\vec{\pi}_{81})
\]
\[
= -1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 1.
\]
Similarly,
\[
N_v(0, 7, 5) = N_v(1, 7, 5) = \ldots = N_v(6, 7, 5) = 1 = \frac{P(5)}{7}.
\]
In general we can write;
\[
N_v(k, 7, 7n + 5) = \frac{P(7n + 5)}{7}; \quad 0 \leq k \leq 6.
\]
Hence the result.

➢ The result is:

$$N_v(k, 1, ln + 6) = \frac{P(1 \ln + 6)}{11}.$$  

Proof: We prove the result with an example. The vector partitions of 6 are given in the table below:

<table>
<thead>
<tr>
<th>Vector partitions of 6</th>
<th>Weight $\omega(\vec{\pi})$</th>
<th>Crank $(\vec{\pi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{\pi}_1 = (\phi, \phi, 6)$</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>$\vec{\pi}_2 = (\phi, \phi, 5 + 1)$</td>
<td>+1</td>
<td>-2</td>
</tr>
<tr>
<td>$\vec{\pi}_3 = (\phi, \phi, 4 + 2)$</td>
<td>+1</td>
<td>-2</td>
</tr>
<tr>
<td>$\vec{\pi}_4 = (\phi, \phi, 4 + 1 + 1)$</td>
<td>+1</td>
<td>-3</td>
</tr>
<tr>
<td>$\vec{\pi}_5 = (\phi, \phi, 3 + 3)$</td>
<td>+1</td>
<td>-2</td>
</tr>
<tr>
<td>$\vec{\pi}_6 = (\phi, \phi, 3 + 2 + 1)$</td>
<td>+1</td>
<td>-3</td>
</tr>
<tr>
<td>$\vec{\pi}_7 = (\phi, \phi, 3 + 1 + 1 + 1)$</td>
<td>+1</td>
<td>-4</td>
</tr>
<tr>
<td>$\vec{\pi}_8 = (\phi, \phi, 2 + 2 + 2)$</td>
<td>+1</td>
<td>-3</td>
</tr>
<tr>
<td>$\vec{\pi}_9 = (\phi, \phi, 2 + 2 + 1 + 1)$</td>
<td>+1</td>
<td>-4</td>
</tr>
<tr>
<td>$\vec{\pi}_{10} = (\phi, \phi, 2 + 1 + 1 + 1 + 1)$</td>
<td>+1</td>
<td>-5</td>
</tr>
<tr>
<td>$\vec{\pi}_{11} = (\phi, \phi, 1 + 1 + 1 + 1 + 1 + 1)$</td>
<td>+1</td>
<td>-6</td>
</tr>
<tr>
<td>$\vec{\pi}_{12} = (\phi, 6, \phi)$</td>
<td>+1</td>
<td>1</td>
</tr>
<tr>
<td>$\vec{\pi}_{13} = (\phi, 5 + 1, \phi)$</td>
<td>+1</td>
<td>2</td>
</tr>
<tr>
<td>$\vec{\pi}_{14} = (\phi, 4 + 2, \phi)$</td>
<td>+1</td>
<td>2</td>
</tr>
<tr>
<td>$\vec{\pi}_{15} = (\phi, 4 + 1 + 1, \phi)$</td>
<td>+1</td>
<td>3</td>
</tr>
<tr>
<td>$\vec{\pi}_{16} = (\phi, 3 + 3, \phi)$</td>
<td>+1</td>
<td>2</td>
</tr>
<tr>
<td>$\vec{\pi}_{17} = (\phi, 3 + 2 + 1, \phi)$</td>
<td>+1</td>
<td>3</td>
</tr>
<tr>
<td>$\vec{\pi}_{18} = (\phi, 3 + 1 + 1 + 1, \phi)$</td>
<td>+1</td>
<td>4</td>
</tr>
<tr>
<td>$\vec{\pi}_{19} = (\phi, 2 + 2 + 2, \phi)$</td>
<td>+1</td>
<td>3</td>
</tr>
<tr>
<td>$\vec{\pi}_{20} = (\phi, 2 + 2 + 1 + 1, \phi)$</td>
<td>+1</td>
<td>4</td>
</tr>
<tr>
<td>$\vec{\pi}_{21} = (\phi, 2 + 1 + 1 + 1 + 1, \phi)$</td>
<td>+1</td>
<td>5</td>
</tr>
<tr>
<td>$\vec{\pi}_{22} = (\phi, 1 + 1 + 1 + 1 + 1 + 1, \phi)$</td>
<td>+1</td>
<td>6</td>
</tr>
<tr>
<td>$\vec{\pi}_{23} = (6, \phi, \phi)$</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ \bar{\pi}_{24} = (5 + 1, \phi, \phi) \]
\[ \bar{\pi}_{25} = (4 + 2, \phi, \phi) \]
\[ \bar{\pi}_{26} = (3 + 2 + 1, \phi, \phi) \]
\[ \bar{\pi}_{27} = (\phi, 5, 1) \]
\[ \bar{\pi}_{28} = (\phi, 1, 5) \]
\[ \bar{\pi}_{29} = (\phi, 4, 2) \]
\[ \bar{\pi}_{30} = (\phi, 2, 4) \]
\[ \bar{\pi}_{31} = (\phi, 4, 1) \]
\[ \bar{\pi}_{32} = (\phi, 4, 1 + 1) \]
\[ \bar{\pi}_{33} = (\phi, 1, 4 + 1) \]
\[ \bar{\pi}_{34} = (\phi, 1 + 1, 4) \]
\[ \bar{\pi}_{35} = (\phi, 3, 3) \]
\[ \bar{\pi}_{36} = (\phi, 3 + 2, 1) \]
\[ \bar{\pi}_{37} = (\phi, 1, 3 + 2) \]
\[ \bar{\pi}_{38} = (\phi, 3, 2 + 1) \]
\[ \bar{\pi}_{39} = (\phi, 2 + 1, 3) \]
\[ \bar{\pi}_{40} = (\phi, 1 + 3, 2) \]
\[ \bar{\pi}_{41} = (\phi, 2, 1 + 3) \]
\[ \bar{\pi}_{42} = (\phi, 3, 1 + 1 + 1) \]
\[ \bar{\pi}_{43} = (\phi, 3 + 1, 1 + 1) \]
\[ \bar{\pi}_{44} = (5, \phi, 1) \]
\[ \bar{\pi}_{45} = (5, 1, \phi) \]
\[ \bar{\pi}_{46} = (4, \phi, 2) \]
\[ \bar{\pi}_{47} = (4, 2, \phi) \]
\[ \bar{\pi}_{48} = (\phi, 1 + 1 + 1, 3) \]
\[ \bar{\pi}_{49} = (\phi, 1 + 1, 3 + 1) \]
\[ \bar{\pi}_{50} = (\phi, 1, 3 + 1 + 1) \]
\[ \bar{\pi}_{51} = (\phi, 3 + 1 + 1, 1) \]
\[ \bar{\pi}_{52} = (\phi, 2 + 2, 2) \]
\[\tilde{\pi}_{53} = (\phi, 2, 2 + 2)\]  
\[\tilde{\pi}_{54} = (\phi, 2, 1 + 1 + 1 + 1)\]  
\[\tilde{\pi}_{55} = (\phi, 1 + 1 + 1 + 1, 2)\]  
\[\tilde{\pi}_{56} = (\phi, 2 + 1, 1 + 1 + 1)\]  
\[\tilde{\pi}_{57} = (\phi, 1 + 1 + 1, 2 + 1)\]  
\[\tilde{\pi}_{58} = (\phi, 2 + 1, 1, 1 + 1)\]  
\[\tilde{\pi}_{59} = (\phi, 1 + 1, 2 + 1 + 1)\]  
\[\tilde{\pi}_{60} = (\phi, 1 + 1 + 1 + 1, 1 + 1)\]  
\[\tilde{\pi}_{61} = (\phi, 1 + 1 + 1, 1 + 1 + 1)\]  
\[\tilde{\pi}_{62} = (\phi, 1 + 1, 1 + 1 + 1, 1)\]  
\[\tilde{\pi}_{63} = (\phi, 1, 1 + 1 + 1 + 1, 1)\]  
\[\tilde{\pi}_{64} = (\phi, 1 + 1 + 1 + 1, 1 + 1, 1)\]  
\[\tilde{\pi}_{65} = (3, 2, 1)\]  
\[\tilde{\pi}_{66} = (3, 1, 2)\]  
\[\tilde{\pi}_{67} = (2, 3, 1)\]  
\[\tilde{\pi}_{68} = (2, 1, 3)\]  
\[\tilde{\pi}_{69} = (1, 2, 3)\]  
\[\tilde{\pi}_{70} = (1, 3, 2)\]  
\[\tilde{\pi}_{71} = (3, 1, 1 + 1)\]  
\[\tilde{\pi}_{72} = (3, 1 + 1, 1)\]  
\[\tilde{\pi}_{73} = (2, 2 + 1, 1)\]  
\[\tilde{\pi}_{74} = (2, 1, 1 + 2)\]  
\[\tilde{\pi}_{75} = (1, 1 + 1 + 1, 1 + 1)\]  
\[\tilde{\pi}_{76} = (1, 1 + 1, 1 + 1 + 1)\]  
\[\tilde{\pi}_{77} = (1, 1, 1 + 1 + 1 + 1)\]  
\[\tilde{\pi}_{78} = (1, 1 + 1 + 1 + 1 + 1)\]  
\[\tilde{\pi}_{79} = (2, 1 + 1, 1 + 1)\]  
\[\tilde{\pi}_{80} = (4, 1, 1)\]  
\[\tilde{\pi}_{81} = (3, \phi, 3)\]
<table>
<thead>
<tr>
<th>( \tilde{\pi} )</th>
<th>( \phi )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\pi}_{82} )</td>
<td>( (3,3,\phi) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{83} )</td>
<td>( (3,1+1+1,\phi) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{84} )</td>
<td>( (3,\phi,1+1+1) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{85} )</td>
<td>( (2,2+2,\phi) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{86} )</td>
<td>( (2,\phi,2+2) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{87} )</td>
<td>( (2,2+1+1,\phi) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{88} )</td>
<td>( (2,\phi,2+1+1) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{89} )</td>
<td>( (2,\phi,2+2) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{90} )</td>
<td>( (2,\phi,1+1+1+1) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{91} )</td>
<td>( (1,1+1+1+1,\phi) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{92} )</td>
<td>( (1,\phi,1+1+1+1) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{93} )</td>
<td>( (1+2,3,\phi) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{94} )</td>
<td>( (1+2,\phi,3) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{95} )</td>
<td>( (3+1,2,\phi) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{96} )</td>
<td>( (3+1,\phi,2) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{97} )</td>
<td>( (3+1,1,1) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{98} )</td>
<td>( (4+1,1,\phi) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{99} )</td>
<td>( (4+1,\phi,1) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{100} )</td>
<td>( (4,1+1,\phi) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{101} )</td>
<td>( (4,\phi,1+1) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{102} )</td>
<td>( (3+1,1+1,\phi) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{103} )</td>
<td>( (3+1,\phi,1+1) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{104} )</td>
<td>( (2+1,1+1+1,\phi) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{105} )</td>
<td>( (2+1,\phi,1+1+1) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{106} )</td>
<td>( (2+1,1,2) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{107} )</td>
<td>( (2+1,2,1) )</td>
<td>+1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{108} )</td>
<td>( (1,2+1,2) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{109} )</td>
<td>( (1,2,2+1) )</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\pi}_{110} )</td>
<td>( (1,2+3,\phi) )</td>
<td>-1</td>
</tr>
</tbody>
</table>
\begin{array}{|c|c|c|}
\hline
\tilde{\pi}_{111} &= (1, \phi, 2 + 3) & -1 & -2 \\
\tilde{\pi}_{112} &= (\phi, 4, 2) & +1 & 1 \\
\tilde{\pi}_{113} &= (2 + 3, \phi, 1) & +1 & -1 \\
\tilde{\pi}_{114} &= (2, 1 + 3, \phi) & -1 & 2 \\
\tilde{\pi}_{115} &= (2, \phi, 3 + 1) & -1 & -2 \\
\tilde{\pi}_{116} &= (1, 2 + 2 + 1, \phi) & -1 & 3 \\
\tilde{\pi}_{117} &= (1, \phi, 2 + 2 + 1) & -1 & -3 \\
\tilde{\pi}_{118} &= (2 + 1, 1 + 1) & +1 & 1 \\
\tilde{\pi}_{119} &= (2 + 1, 1 + 1 + 1) & +1 & -1 \\
\tilde{\pi}_{120} &= (1, 1 + 2 + 1) & -1 & 0 \\
\tilde{\pi}_{121} &= (1, 2 + 1, 1 + 1) & -1 & 0 \\
\hline
\end{array}

From this table we have;

\[
N_V(0, 11, 6) = \omega(\tilde{\pi}_{23}) + \omega(\tilde{\pi}_{24}) + \omega(\tilde{\pi}_{25}) + \omega(\tilde{\pi}_{26}) + \\
\omega(\tilde{\pi}_{27}) + \omega(\tilde{\pi}_{28}) + \omega(\tilde{\pi}_{29}) + \omega(\tilde{\pi}_{30}) + \omega(\tilde{\pi}_{35}) + \\
\omega(\tilde{\pi}_{43}) + \omega(\tilde{\pi}_{49}) + \omega(\tilde{\pi}_{62}) + \omega(\tilde{\pi}_{65}) + \omega(\tilde{\pi}_{66}) + \omega(\tilde{\pi}_{67}) + \omega(\tilde{\pi}_{68}) + \omega(\tilde{\pi}_{69}) + \\
\omega(\tilde{\pi}_{70}) + \omega(\tilde{\pi}_{79}) + \omega(\tilde{\pi}_{82}) + \omega(\tilde{\pi}_{85}) + \omega(\tilde{\pi}_{86}) + \omega(\tilde{\pi}_{106}) + \omega(\tilde{\pi}_{107}) + \omega(\tilde{\pi}_{120}) + \omega(\tilde{\pi}_{121}) \\
= -1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 + 1 + 1 + 1 - 1 - 1 = 1.
\]

\[
N_V(0, 11, 6) = 1 = \frac{P(6)}{11}, \text{ where } n = 0 \text{ and } k = 0.
\]

Hence the result.

**Conclusions**

We verified that for any positive integral value of \( n \) in the relation \( P(n) = \sum_{m=-\infty}^{\infty} N_V(m, n) \)
and easily can find generating function for \( N_V(m, n) \) in terms of various corresponding cranks of vector partitions.

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REFERENCES


