New Basis Functions for Model Reduction of Nonlinear PDEs

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Abstract—The selection of the spatial basis functions is very important for model reduction of the nonlinear partial differential equations (PDEs) under time/space separation framework, which will significantly affect the accuracy and efficiency of the modeling. Using the spatial basis functions expansions and the Galerkin method, the finite-dimensional ordinary differential equation (ODE) systems can be obtained from the PDEs. However, the general basis functions are not optimal in the sense that the dimensions of the ODE system are not lowest at a given modeling accuracy. The current study proposes new basis functions for the model reduction of nonlinear PDEs, which are obtained by linear combinations of general spatial basis functions. The transformation matrix for the combination coefficients is derived from straightforward optimization techniques for an improved spatio-temporal error function between the approximation and the measured spatial-temporal output. The derivation of the improved error functions also considers the influence of the variance of the spatial-temporal error. Using the new basis functions expansions and Galerkin method, it can provide a lower dimensional and more precise ODE to approximate the dynamics of the nonlinear PDEs. The modeling performance are compared with the method proposed in reference, and the simulations shows the feasibility and effectiveness of the proposed new basis functions for model reduction of nonlinear PDEs.

Index Terms—Nonlinear PDEs; Dynamical Systems; Basis Functions; Transformation Matrix; Optimization

I. INTRODUCTION

Obtaining a low-dimensional model for the control of industrial processes at a reasonable cost and accuracy is very important in science research and engineering applications. The industrial processes can be described by nonlinear partial differential equations (PDEs) [1, 2, 3] through a process of mathematical first-principle modeling, because their input, output, states and even parameters vary temporally and spatially. In the note, we start out with the kind of nonlinear partial differential equations as follows.

\[
\frac{\partial X}{\partial t} = AX + BU + F(X, \frac{\partial X}{\partial t}, \ldots, z, t) \tag{1}
\]

where \(X(z, t)\) and \(U(z, t)\) denote the vectors of spatio-temporal state variable and manipulated spatio-temporal input respectively. \(z \in [\alpha, \beta] \subset \Omega\) is the spatial coordinate variable, \(t \in [0, \infty)\) is the time variable. \(A\) and \(B\) are two linear operators involve linear spatial derivatives on the state variable and input respectively. \(F(X, \frac{\partial X}{\partial t}, \ldots, z, t)\) is a nonlinear function which contains spatial derivatives of the state variable.

Nonlinear PDE (1) will satisfy many types of the boundary conditions. It can use the following descriptions to denote boundary conditions of the Eq. (1).

\[
b_1 X(\alpha, t) + c_1 \frac{\partial X(\alpha, t)}{\partial z} = d_1 \tag{2}
\]

\[
b_2 X(\beta, t) + c_2 \frac{\partial X(\beta, t)}{\partial z} = d_2
\]

The initial conditions are also given:

\[
X(z, 0) = X_0(z) \tag{3}
\]

where \(b_1, c_1, d_1, b_2, c_2, d_2\) are the corresponding parameters and \(X_0(z)\) is a function of the spatial coordinate variable \(z\).

To precisely characterize the nonlinear PDE systems considered in Eq. (1), the PDE of Eq. (1) are formulated as an infinite-dimensional system in the Hilbert space \(\Lambda\) with the following inner product:

\[
(w_1, w_2) = \int_{\alpha}^{\beta} w_1(z) w_2(z) dz \tag{4}
\]

where \(w_1, w_2\) are two arbitrary functions of Hilbert space \(\Lambda\).

In terms of the suitable choice of spatial basis functions, the nonlinear PDE (1) can be reduced to a finite-dimensional ordinary differential equation (ODE) system by Galerkin method, which is called model reduction for PDEs. A number of model reduction approaches for nonlinear PDEs have already been studied [4, 5, 6]. They result in the feasible implementation of control for practical applications in industrial processes. Conventional approaches based on the time/space discretization, such as the finite difference method and finite element method, can easily transform an infinite-dimensional PDE to a finite-dimensional ODE system. Though they can be used for the model reduction of the PDEs with any complex boundary conditions, however, they often yield high-dimensional ODEs or DE dynamical systems that are unsuitable for synthesizing implementation control design.
Recently, the advanced model reduction approaches are among the most extensively used methods for the model reduction of PDEs. The procedures of the approaches are using the spatial basis functions expansions combined with weighted residual methods [1]. The accuracy and efficiency of the obtained ODE model are mainly dependent on the proper choice of basis functions. The popular basis functions generally used in advanced method are not optimal because the model dimension cannot be minimized for the desired modeling accuracy.

To improve the model performance of the general spatial basis functions, some approaches for obtaining new spatial basis functions are developed for the model reduction of nonlinear PDEs. It assumes that the spatial basis functions used for the model reduction should be taken into account the dynamics of the nonlinear PDEs. The first methodology taking into account the dynamics of the reduced finite-dimensional ODE model to define the basic spatial patterns is called principal interaction patterns (PIPs), which was first introduced by Hasselmann [7] and a general optimization calculation algorithm for reducing a high-dimensional model to a low-dimensional ODE system was derived by Kwasiok [8-10]. However, the variational principle in the calculation optimization algorithm for PIPs requires high dimensional nonlinear minimization steps, so it is not a proper choice to derive the new spatial basis functions for the model reduction of nonlinear PDEs.

To reduce the calculation difficulty of the PIPs, Deng [4] proposed a straightforward algorithm to derive new spatial basis functions. Like PIPs, the new spatial basis functions are linear combinations of general spatial basis functions, and the transformation matrix is derived from balanced truncation method [4] for a corresponding linear ODE system of nonlinear PDEs. Following the above methodology, Jiang and Deng [5] proposed a new algorithm to derive new spatial basis functions by linear transformation, and the transformation matrix is derived from optimization techniques of a spatial-temporal error functions. However, its objective functions of the transformation matrix for optimization calculations can only measure the spatially and temporally integrated squared magnitude between the spatio-temporal approximations based on initial high-order spectral based model with that based on the reduced model.

To derive a low-dimensional ODE system to simply the control design and the parameter settings from the nonlinear PDEs, the current study proposes a set of new basis functions for model reduction of nonlinear PDEs. The new basis functions are also derived by linear combinations of general spatial basis functions, and the transformation matrix is derived from optimization for an improved error function, which measures the spatial-temporal squared error between the measured output and the spatial-temporal approximation based on the new spatial basis functions. To improve the modeling performance based on the new spatial basis functions, the derivation of the improved error functions also considers the influence of the variance of the spatio-temporal error between approximation and the measured output of nonlinear PDEs. After the computations from the optimization for the improved error function, new spatial functions used for the model reduction of nonlinear PDEs can be derived by basis functions transformation. Using the expansions based on the new spatial basis functions and the truncation by Galerkin method, it can provide a more precise ordinary differential equation system with the same modes to approximate the dynamics of the nonlinear PDEs. A numerical example is used to demonstrate the effectiveness and performance of the proposed new spatial basis functions for the model reduction of the nonlinear PDEs.

II. EIGENFUNCTIONS FOR MODEL REDUCTION OF PDEs

Spatial basis functions expansions combined with weighted residual methods (WRM) are among the most extensively used methods for the model reduction of nonlinear PDEs. It has been shown to provide very accurate approximations of sufficiently smooth solutions. In recent years, spatial basis functions expansions have become widespread used for model reduction in various fields including fluid dynamics, quantum mechanics, heat conduction and weather prediction.

It is well known that a continuous function can be approximated using Fourier series expansions. Based on this principle, the spatio-temporal variables \( X(z,t) \) and \( U(z,t) \) of the PDE (1) can be expanded by a set of spatial trial functions \( \{\phi_i(z)\}_{i=1}^{\infty} \) as follows:

\[
X(z,t) = \sum_{i=1}^{\infty} x_i(t) \phi_i(z) \tag{5}
\]

\[
U(z,t) = \sum_{i=1}^{\infty} u_i(t) \phi_i(z) \tag{6}
\]

where \( x_i(t) \), \( u_i(t) \) denote the corresponding time coefficient of \( \phi_i(z) \), \( \{\phi_i(z)\}_{i=1}^{\infty} \) denotes the infinite set of the eigenfunctions of linear operator used for the model reduction of PDE (1). Similar to the Fourier series, the spatio-temporal variable is separated into a set of spatial BF and the temporal variables. In practice, only the first \( N \) modes of the expansion (5) will be retained as follows.

\[
X_N(z,t) = \sum_{i=1}^{N} x_i(t) \phi_i(z) \tag{7}
\]

In the WRM, the residual equation of the model (1) generated from the truncated expansion (7) can be expressed as

\[
R_N = \frac{\partial X_N(z,t)}{\partial t} - (AX_N(z,t) + BU(z,t)) + F\left(X_N(z,t), \frac{\partial X_N(z,t)}{\partial z}, \ldots, z, t\right) \tag{8}
\]

which is made in the sense that
\[(R_{\phi}, \phi) = 0, \ i = 1,2,\ldots, N \quad (9)\]

where \(\{\phi(z)\}_{i=1}^{N}\) is a set of weighting functions to be chosen. If the weighting functions \(\{\phi(z)\}_{i=1}^{N}\) are chosen to be the same as basis functions \(\{\hat{\phi}(z)\}_{i=1}^{N}\), the method is called the Galerkin Method. This is an easy way to obtain an ODE model from Eq. (9) for the time evolution of the expansion coefficients. Using the Galerkin method, the following ODE model is obtained.

\[
\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t) + f(x(t), t) \quad (10)
\]

where \(x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T\);
\[
\tilde{A} = \text{diag}(-\lambda_1, -\lambda_2, \ldots, -\lambda_N) ;
\]
\[
\tilde{B} = [\tilde{b}_{11}, \tilde{b}_{21}, \ldots, \tilde{b}_{N1} ]^T = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{21} & \ldots & \tilde{b}_{N1} \\ \tilde{b}_{12} & \tilde{b}_{22} & \ldots & \tilde{b}_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{1m} & \tilde{b}_{2m} & \ldots & \tilde{b}_{Nm} \end{bmatrix} ;
\]
\[
\lambda_i, i = 1,2,\ldots, N \text{ denote the eigenvalues of the spatial operator } A \ . \ \tilde{b}_{ij} \text{ is the constants representing the scalar distribution of control input, where}
\]
\[
\tilde{b}_{ij} = \int_{\Omega} b(\phi_j(z))\phi_i(z)dz \quad (11)
\]
\[
f(x(t), t) \text{ in Eq.}(10) \text{ denotes the nonlinear terms, where}
\]
\[
f_i(x(t), t) = \int_{\Omega} F(X_N(z,t), \partial X_N(z,t))/\partial z, \ldots, t, \partial z)\phi_i(z)dz \quad (12)
\]

III. NEW BASIS FUNCTIONS BY LINEAR TRANSFORMATION

New spatial orthogonal basis functions for time/ space variables separation have to be derived to obtain lower dimensional and more precise approximation models. Let each new spatial basis functions be a linear combination of the eigenfunctions of nonlinear PDEs [5]. Defining a basis function transformation matrix \(S\), the following is obtained:

\[
\{\psi_1(z), \psi_2(z), \ldots, \psi_N(z)\} = \{\phi_1(z), \phi_2(z), \ldots, \phi_N(z)\}S \quad (13)
\]

where \(k < N\), it means that the number of new basis functions should be smaller than eigenfunctions. \(\psi(z)\) and \(\phi(z)\) denote new basis functions and initial basis functions respectively.

After the time/space separation based on new spatial basis functions (13), a new lower-dimensional ODE system can be derived.

\[
\dot{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) + \int \dot{f}(\hat{x}(t), t) \quad (14)
\]

where \(\hat{x}(t) = [\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_N(t)]^T\);
\[
\hat{A} = S^T\tilde{A}S , \hat{B} = S^T\tilde{B} ;
\]
\[
\hat{f}(\hat{x}(t), t) = [\hat{f}_1(\hat{x}(t), t), \hat{f}_2(\hat{x}(t), t), \ldots, \hat{f}_N(\hat{x}(t), t)]
\]

and \(\hat{f}_i(\hat{x}(t), t) = \int_{\Omega} F(X_N(z,t), \partial X_N(z,t))/\partial z, \ldots, t)\psi_i(z)dz \)

Based on the new basis functions (13), the spectral-based model Eq. (10), and the new system Eq. (14), the equation (13) can be derived as follows:

\[
x(t) = \hat{S}\hat{x}(t) \text{ or } \hat{x}(t) = S^T x(t) \quad (15)
\]

There are many calculation methods of transformation matrix \(S\), such as balanced truncation method [4, 11] and optimization method [5]. For the optimization method [12], the objective functions is given as follows

\[
\text{Error} = \int_{0}^{t_{\text{max}}} \left( \sum_{i=1}^{N} x_i(t)\phi_i(z) \right)^2 dz dt
\]

where \([0, t_{\text{max}}]\) is the integrating time interval, \(x(t)\) and \(\dot{x}(t)\) are the expansions coefficients in the ODE model (10) and (14), respectively.

Moreover, \(\sum_{i=1}^{N} x_i(t)\phi_i(z)\) denotes the spatial-temporal approximation of the initial ODE model (10), \(\sum_{i=1}^{N} \dot{x}_i(t)\psi_i(z)\) denotes the spatial- temporal approximation of the reduced ODE model (14). The optimal set of the basis functions transformation matrix is determined by minimizing the error function (16). However, the objective functions (16) for optimization only can measure the spatially and temporally integrated squared error between the approximations based on initial high-order spectral based model (10) with that based on the reduced model (14). For this reason, the new basis functions derived by optimization for the error function (16) is not optimal, and the precision of the model reduction for nonlinear PDEs will not be high enough.

IV. OPTIMIZATION FOR THE IMPROVED ERROR FUNCTION

An improved error function that measures the spatially and temporally integrated squared error between the solution of the nonlinear PDE and the approximation of the reduced model based on the new spatial basis functions is presented in this section. Let \(X_n(z,t)\) be the solution of the nonlinear PDE (1), the spatial-temporal error between the solution of the nonlinear PDE and the approximation of reduced model (14) based on the new spatial basis functions is introduced as follows.

\[
\text{Error} = \int_{0}^{t_{\text{max}}} \left( \sum_{i=1}^{N} x_i(t)\phi_i(z) \right)^2 dz dt
\]
\[ e(z,t) = X_m(x,t) - X_{w_0}(z,t) = X_m(x,t) - \sum_{j=1}^{\infty} \varphi_j(z) \phi_j(z) \] (17)

Combining (13) and (15), the following error function can be derived.

\[ e(z,t) = X_m(x,t) - \left[ x_1(t), x_2(t), \ldots, x_n(t) \right] S S^{-1} \left[ \phi_1(z), \phi_2(z), \ldots, \phi_n(z) \right]^T \] (18)

The objective function of optimization for spatial basis functions transformation matrix is introduced as follows.

\[ \text{Error} = \int_0^{t_{\text{max}}} \int_{\Omega} \left( e(z,t) \right)^2 dz dt + \sum_{i=1}^{\infty} \text{Var} \left( e(z,t) \right) dt \] (19)

where the \([0,t_{\text{max}}]\) is the integrating time interval, \(\rho\) is the ratio that determined by the order of magnitude in the two terms of (19), \(\text{Var} \) is given as follows.

\[ \text{Var}(e(z,t)) = \int_{t_{\text{max}}}^{t_{\text{max}}} \int_{\Omega} \left( e(z,t) - \overline{e(x,t)} \right)^2 dz dt \] (20)

\[ \overline{e(x,t)} = \int_0^{t_{\text{max}}} \int_{\Omega} e(z,t) dz dt \] (21)

The optimal set of the basis function transformation matrix \(R\) is determined by minimizing the error function (19).

The integral in Eq. (18) must be approximated by a finite sum to evaluate the error function. Let the \(X_m(z,t)\) is measured at the \(n\) spatial locations \(z_1, z_2, \ldots, z_n\) and some sampling time \(t_1, t_2, \ldots, t_{\text{max}}\). The spatial-temporal error \(e(z,t)\) in eq. (18) can be expressed by a matrix \(E_{\text{maxx}}\) as Eq. (22).

\[ E_{\text{maxx}} = \begin{bmatrix} X_{z_1}(t_1) & X_{z_1}(t_2) & \cdots & X_{z_1}(t_{\text{max}}) \\ X_{z_2}(t_1) & X_{z_2}(t_2) & \cdots & X_{z_2}(t_{\text{max}}) \\ \vdots & \vdots & \ddots & \vdots \\ X_{z_n}(t_1) & X_{z_n}(t_2) & \cdots & X_{z_n}(t_{\text{max}}) \end{bmatrix} \]

\[ \begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_{\text{max}}) \\ x_2(t_1) & x_2(t_2) & \cdots & x_2(t_{\text{max}}) \\ \vdots & \vdots & \ddots & \vdots \\ x_n(t_1) & x_n(t_2) & \cdots & x_n(t_{\text{max}}) \end{bmatrix} \]

The optimization of the basis functions transformation matrix then becomes

\[ \text{Error} = \frac{1}{N_{\text{tim}} \cdot n} \sum_{i=1}^{n} \sum_{j=1}^{N_{\text{tim}}} (E_{ij})^2 + \rho (\text{Var}(E))^2 \] (23)

where \(t_i\) are equally spaced mesh points in the interval \([0,t_{\text{max}}]\). The minimization of the eq.(23) in the current study is related to the solution of the initial model (10) and the measured spatial-temporal state of the nonlinear PDE(1). Generally, the minimization of the Eq.(23) poses a nonlinear optimization problem, which is usually solved numerically by iterative techniques. However, the iterative techniques are more expensive computationally for a high-dimensional nonlinear optimization, which can be greatly affected by the choice of the initial values in the process of searching for an optimum point of the error function in a descent direction. Thus, a stochastic optimization method is recommended to calculate the numerically for Eq.(23) in the current research.

As a stochastic optimization method, particle swarm optimization (PSO) is a recently invented high performance algorithm, which is introduced in the middle of the 1990s [13-15]. PSO is an efficient, robust and simple optimization algorithm for solving many optimization problems. In nature, PSO maintains a swarm of candidate solutions, referred to as particles. The method is inspired by the movement of particles and their interactions with their neighbors in the group. The principle and details of PSO algorithm is given in Ref [15].

The main steps of the algorithm used for optimization of basis functions transformation matrix are outlined as follows:

1. The \(X_m(z,t)\) is measured at the \(n\) spatial locations \(z_1, z_2, \ldots, z_n\) and some sampling time \(t_1, t_2, \ldots, t_{\text{max}}\). It should be saved to use for the optimization of the Eq. (23).
2. Calculate the value of the initial basis functions \(\phi_i(z)\) at the \(n\) spatial locations \(z_1, z_2, \ldots, z_n\).
3. Given the initial condition, \(x(t)\) are calculated using the ODE (Eq.(10)) by fourth-order Runge-Kutta method.
4. Given \(k\), an initial matrix \(\tilde{S}\) which is not an orthogonal, is calculated using Eq.(23) by PSO. The ratio \(\rho\) is determined by the order of magnitudes of the two terms of the error functions (23).
5. The singular value decomposition of the initial \(\tilde{S}\) in Step 4 is calculated as \(\tilde{S} = U W V^T\), where \(U\) and \(V\) are two orthogonal matrixes of eigenvectors of \(\tilde{S} \tilde{S}^T\) and \(\tilde{S}^T \tilde{S}\), respectively. The size of the matrix \(W\) is \(N \times k\) and only its main diagonal has non-zero elements which are the singular values of \(\tilde{S}\) sorted in descending order.
6. The basis functions transformation matrix \(S = U W\) is obtained. The orthogonality of \(S\) is satisfied by the orthogonal matrix \(U\) and \(W\).

It can be derived that

\[ S S^{-1} = U W (U W)^T = U W (W^T W)^{-1} W^T U^T = \tilde{S} \tilde{S}^{-1} \] (24)

which means that the minimum of the error function (23) from Eq. (18).

V. NUMERICAL EXAMPLE

To evaluate the proposed the new basis functions for model reduction of nonlinear partial differential equations,
a typical nonlinear PDE that depicts a distributed process including a catalytic rod is studied. After choosing initial spatial orthogonal basis functions for time/space variables separation and Galerkin truncation, the nonlinear PDE is reduced to a finite-dimensional ODE system, which can approximate the dynamic of the PDE. The results of performance for model reduction of the example using the basis functions proposed in Ref. [5] are given for comparison.

The root-mean-square error (RMSE) is defined as the performance index, whereas \( y(x,t) \) denotes the spatial temporal error between the real dynamical process and the approximation model.

\[
RMSE = \left( \int \sum y(x,t)^2 \, dz / \int dz \sum \Delta t \right)^{1/2}
\]

where \( \Delta t \) denotes the time sampling interval. For easy RMSE comparison, this error value indicates the spatio-temporal approximation on a series of sampling points for nonlinear PDE.

The mathematical model that describes the spatio-temporal evolution of the rod temperature with weak assumptions consists of the following PDE [3]:

\[
\frac{\partial X(z,t)}{\partial t} = \frac{\partial^2 X(z,t)}{\partial z^2} + \beta_1 \left( e^{-\gamma(1-x)} - e^{-x} \right) + \beta_2 \left( h(z)^T u(t) - X(z,t) \right)
\]

where \( X(z,t) \) represents the spatio-temporal coupling variables, and (26) is subject to the Dirichlet boundary and initial conditions.

\[
X(0,t) = 0, \quad X(\pi,t) = 0, \quad X(z,0) = X_0(z)
\]

Four actuators \( u(t) = [u_1(t), \ldots, u_4(t)]^T \) are available with the spatial distribution functions \( h(z) = [h_1(z), \ldots, h_4(z)]^T \), where

\[
h_i(z) = H(z - \frac{\gamma}{4}) - H(z - \frac{\gamma}{4})
\]

where \( i = 1, 2, \ldots, 4 \)

and \( H(\cdot) \) is the standard Heaviside function. The process parameters are often set as \( \beta_1 = 50, \beta_2 = 2, \gamma = 4 \). In this case, the input signals are selected as follows [20].

\[
u_i(t) = 1.1 + 5 \sin \left( t / 10 + i / 10 \right), \quad i = 1, \ldots, 4
\]

Suppose that 41 sensors uniformly distributed in the space are used for measurement. Three hundred data for each sensor locations are collected from (26). The sampling interval \( \Delta t \) is 0.01s and the simulation time is 3s. The initial condition \( X_0(z) \) is set to be \( \sin z \).

A spectral-based model is developed for Eq. (26), where the family of spatial orthogonal basis functions, \( \sqrt{2 / \pi} \sin(kz), \quad k = 1, 2, \ldots, \infty \) are used for time/space separation and finite-dimensional truncation. The new basis functions are obtained though basis functions transformation, where the transformation matrix is derived from optimization for the proposed improved error function. Each new spatial basis functions is a linear combination of the spectral eigenfunctions.

### A. Comparison with the Basis Functions Proposed in Ref. [5]

For comparison the performance with the basis functions proposed in Ref. [5], the optimal combinations of spatial basis functions are derived by basis functions transformation, where the transformation matrix is obtained from the optimization of the error function (16). Similarly, the optimization of error functions (23) can be implemented using the algorithm proposed in this note. The ratio \( \rho \) in the optimization process is set to be 20, which is the proportionality estimated for the two terms of the error functions (23). Synthesis of spatial variables and temporal output of dynamical ODE system, the RMSEs over the testing data are compared in Table 1.

To illustrate the effectiveness of the proposed new basis functions, the RMSEs over the testing data based on initial spectral basis functions are computed, which are shown in Table 2. It is obvious that the RMSE based on new basis functions from the optimization of improved error function (23) is less than that based on the same order basis function in Ref.[5]. Moreover, the RMSE based on three new basis functions from the optimization of improved error function (23) is less than that based on the eight spectral spatial basis functions. It means that the model based on the new spatial basis functions from the optimization of improved error function (23) can capture the dominant dynamics of the nonlinear PDEs with much fewer modes.

### TABLE I. RMSEs of Two Kinds of Basis Functions Over the Testing Data

<table>
<thead>
<tr>
<th>RMSE</th>
<th>1 modes</th>
<th>2 modes</th>
<th>3 modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis functions proposed in Ref. [5]</td>
<td>0.1509</td>
<td>0.15583</td>
<td>0.13646</td>
</tr>
<tr>
<td>Basis functions proposed in this note</td>
<td>0.1193</td>
<td>0.11528</td>
<td>0.0836</td>
</tr>
</tbody>
</table>

### TABLE II. RMSE Based on the Spectral Basis Functions

<table>
<thead>
<tr>
<th>RMSE</th>
<th>1 modes</th>
<th>2 modes</th>
<th>3 modes</th>
<th>4 modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8494</td>
<td>0.8288</td>
<td>0.2084</td>
<td>0.2060</td>
<td></td>
</tr>
<tr>
<td>0.1357</td>
<td>0.1353</td>
<td>0.1264</td>
<td>0.1263</td>
<td></td>
</tr>
</tbody>
</table>

### B. Low-dimensional Approximation Based on Three Kinds of Spatial Basis Functions

The newly derived model up to three modes is sufficient to capture the dynamics of the nonlinear PDEs for control design. The performance of the reduced model based on three new spatial basis functions is illustrated respectively.

The three new spatial basis functions derived from the error function (16) and improved error function (23) are
shown in Fig. 1 and Fig. 2, respectively. The corresponding same-order spectral basis functions are shown in Fig. 3.

Figure 1. The first three new basis functions proposed in this note

Figure 2. The first three new basis functions proposed in the Ref. [4]

Figure 3. The first three spectral basis functions

A set of 300 data is collected for testing to demonstrate the spatial-temporal performance of the two kinds of basis functions (Fig. 4).

The spatio-temporal approximated output of the models combined with three kinds of spatial basis functions on the testing data are shown in Fig.5, Fig.6 and Fig.7, while the spatio-temporal errors of the prediction output of models combined with three kinds of spatial basis functions are shown in Fig.8, Fig.9 and Fig.10, which RMSEs are 0.2084, 0.1365 and 0.0836, respectively.

Figure 4. The measured output for testing

Figure 5. The approximated output based on three spectral basis functions

Figure 6. The approximated output based on three new basis functions derived from the error function (16)

The results for the comparisons of the distributed predicted error have shown that the modeling performance based on the new spatial basis functions in this note is superior to the new spatial basis functions in Ref. [4] and general spatial basis functions in spectral method.
After the spatial basis functions expansions based on the new basis functions and Galerkin truncation, it can provide a lower dimensional and more precise ordinary differential equation system to approximate the dynamics of the nonlinear PDEs.

VI. CONCLUSIONS

The current study proposes a set of new basis functions for model reduction of nonlinear PDEs, which can be obtained by linear combinations of general spatial basis functions. The transformation matrix for the combination coefficients is derived from straightforward optimization techniques for a proposed improved error function, which measures the spatial-temporal squared error between the approximations and the measured spatial-temporal outputs. The derivation of the improved error functions also considers the influence of the variance of the error between approximation and the measured output. Using the new spatial basis functions, it can provide a lower dimensional and more precise ordinary differential equation system to approximate the dynamics of the nonlinear PDEs. The numerical example shows the feasibility and effectiveness of the set of new spatial basis functions for model reduction of nonlinear PDEs.

ACKNOWLEDGEMENT

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