Adaptive Boundary Elements and Error Estimation for Elastic Problems

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Abstract—In traditional thinking, when the elastic problems are solved, we need to repeatedly plot element grids and analyze computing results according to diverse precision requirement. Against the malpractice exists in the above process, a new method of error estimation was suggested on H-R adaptive boundary element method in this paper. Based on the discrete meshes that are generated for the process of H-R adaptive refinement, the solution error was estimated by the interpolation residue. In addition, this method is easy to programming, which is carried out in the program by automatically creating new adaptive data files. Then a great deal of fore-disposal and post-disposal can be saved. Its validity and effectiveness have been confirmed by numerical example.

Index Terms—H-R Adaptive Boundary Elements; Interpolation Residue; Error Estimation

I. INTRODUCTION

The boundary element method is a powerful tool of computational mechanics and computational mathematics. Compared with the finite element method, the dimension reduction, Construction of boundary element discrete grid is relatively simple. In order to improve the computational efficiency and avoid subjective mistakes, people developed the adaptive boundary element method, the basic idea is to use computer to automatically judge and improve the accuracy of the boundary element solution. When analyzed by boundary element program, the analysts only need to define a discrete grid which can reflect the boundary geometric features, and specify the accuracy that the boundary element solution to achieve. The computer will automatically begin an iterative process to generate an optimization discrete mesh and it meets the accuracy requirements. In this process, the boundary element approximation solution error estimation is the most important part, because it is used to determine the boundary element approximation accuracy, also used for adaptive subdivision process driving the discrete grid. Therefore, the boundary element solution error estimation method has become one of the research focus in the field of adaptive boundary element method.

In the sense of Galerkin, calculate numerical solution of boundary integral equation that the boundary has singular point, because near singular points, the change of unknown variables is rapid. It is need for boundary unit subdivision which closes to a singularity. Adaptive boundary element method to deal with such problems is an effective method. This method requires that the error between exact solution and the approximate solution obtained by the boundary element method and distribution situation have an accurate grasp and it can be calculated. Typically, error indicators of the local error obtained from the numerical solution, and error estimates in some norm, and this estimate is only determined by the input data and the numerical solution. It can be calculated and be able to represent the local error indicator, which is the so-called a posteriori error estimates. The use of a posteriori error estimates, if found that near the singular point error indicators on the unit does not meet the accuracy requirements set, in the calculation automatically subdivide these units, and so forth, until the local error indicators and error estimators meet the overall set accuracy. It can be said, to achieve a posteriori error an estimate is a prerequisite to adaptive boundary element method to calculate.

Boundary element solution accuracy can be improved by the following processes: 1) H- process[1], larger unit division, the number of units in the grid increases while the order of interpolation function unchanged; 2) P-process [2], increase the order of the interpolation function in the larger error unit, but the total number of unit in the grid unchanged; 3) R- processes[3], rearrange the distribution point position in the larger error unit; 4) the combined process of above methods[4-7], such as HP- process, HR- process. In the adaptive analysis process the optimal mesh is not average assigned the unit to the boundary, but according to the situation of the error analysis, this is more actual.

There are many boundary element approximation error estimation methods, integrated can be summarized into three categories:

1) the residual analysis method: substituted the boundary element approximation solution in boundary integral equation, and use the residual of boundary integral equation for boundary element approximation error measure;
2) improved analytical method: after processing boundary element approximate solutions, in order to get a
more accurate solution, and use the improved solutions to replace the real solution for error analysis;

(3) the sensitivity analysis of solution: by moving the point position, calculating the boundary element solution sensitivity, and use it as the boundary element approximation error measure.

The research shows that [8], if properly handled, the above three methods have good accuracy. However, for the elastostatic problems in general, the residual analysis method is that through the boundary integral equation, link the displacement error and force error in the whole, so the processing is simple. In view of the characteristics of H-R adaptive boundary element method, through the analysis of interpolation error of gradient discrete grid generated in the process of H-R adaptive, then use it as the basis of the boundary element solution error. Major works include: error analysis of initial discrete grid for the HR process which adaptive BEM; accurately delineate the severe degree of boundary displacement and stress changes; predict the location of future subdivision. This means that if properly handled feedback information which is generated by the discrete grid of HR subdivision process, it can be used to estimate error distribution of the boundary element solution on the boundary of the.

II. ADAPTIVE BOUNDARY ELEMENTS

A. The Basic Formula of Adaptive Boundary Element

When we does not consider the body force, boundary element equations of plane elasticity problems are as following:

\[ c_i^T u_i + \int_{\Gamma} p_i^T u_i d\Gamma = \int_{\Gamma} u_i^T p_i d\Gamma \quad (i,k = 1,2) \]  
(1)

where \( u_i \) and \( p_i \) is Kelvin basic solution, represent the displacement and the traction on the boundary; \( c_i \) is the constant term, and it depends on the node location and geometry. Specific selection is as follows:

Points \( i \) in the domain

\[ c'_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  
(2)

Point \( i \) in a smooth boundary

\[ c'_i = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \]  
(3)

Point \( i \) in the corner

\[ c'_i = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \]  
(4)

Discrete the formula (1), and the equivalent matrix equations are obtained

\[ HU = GP \]  
(5)

Among them: \( H \) is \( 2N \times 2N \) matrix; \( G \) is \( 2N \times 3N \) matrix.

With the H-R adaptive boundary element method for solving the boundary integral equation, the initial mesh

with few elements to represent the geometric boundary condition, the unit is used as the grass-roots unit of the adaptive process. Geometric features of boundary element remain unchanged in subdivision process, two times interpolation shape function \( \varphi_i(\xi) \) and the coordinates \( x'_i \) of the nodes can be expressed as

\[ x'_i(\xi) = \sum_{j=1}^3 \varphi_j(\xi)x'_j \quad i = 1, 2 \]  
(6)

That is

\[ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_1 & 0 \\ 0 & 0 & \phi_1 \end{pmatrix} \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix} = \phi^T u \]  
(7)

\[ p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_1 & 0 \\ 0 & 0 & \phi_1 \end{pmatrix} \begin{pmatrix} p_1^T \\ p_2^T \\ p_3^T \end{pmatrix} = \phi^T p \]  
(8)

The quadratic interpolation shape function is

\[ \varphi_1(\xi) = -\frac{1}{2} \xi(\xi - 1) \]
\[ \varphi_2(\xi) = 1 - \xi^2 \]
\[ \varphi_3(\xi) = \frac{1}{2} \xi(\xi + 1) \]

As shown in Figure 1 and Figure 2, for the discrete boundary integral equations, the physical quantity approximation order and the interpolation order of the
were displacement and surface forces of node.

\[ \mathbf{c'}u' + \sum_{j=1}^{NE} \left( \int_{\Gamma_j} p_j \phi d\Gamma \right) u_j' = \sum_{j=1}^{NE} \left( \int_{\Gamma_j} u_j' \phi d\Gamma \right) p_j' \]  

(10)

In the formula, \( NE \) is the total number of cells, \( u_j' \) and \( p_j' \) were displacement and surface forces of node. Along \( d\Gamma \) with the integration, overall coordinate should be changed by unit 1 coordinate, so unit integral in formula (10) should become the following formula

\[ \int_{\Gamma_j} p_j \phi d\Gamma = \int_{\Gamma_j} p_j' \phi [G] d\xi \]

\[ \int_{\Gamma_j} u_j' \phi d\Gamma = \int_{\Gamma_j} u_j' \phi [K] d\xi \]  

(11)

Among them, the jacobian \([G]\) is

\[ |G| = \left[ \left( \frac{dx_1}{d\xi} \right)^2 + \left( \frac{dx_2}{d\xi} \right)^2 \right]^{1/2} \]  

(12)

C. Calculation of Coefficient Matrix

The formula (10), using the numerical integral formula, we have

\[ \mathbf{c'}u' + \sum_{j=1}^{NE} \left( \int_{\Gamma_j} p_j' \phi \right) \mathbf{G}_{ij} u_j' = \sum_{j=1}^{NE} \left( \int_{\Gamma_j} u_j' \phi \right) \mathbf{G}_{ij} p_j' \]  

(13)

In the formula, \( \Gamma_j \) is the number of integral point on unit, \( \phi_x \) is the weight coefficient of integral point, \( p_j' \), \( u_j' \phi \) must be expressed by the variables of integral point. formula (13) is at a source point \( i \), After integrating, we got

\[ \mathbf{c'}u' + \sum_{j=1}^{N} \mathbf{H}_{ij} u_j' = \sum_{j=1}^{N} \mathbf{G}_{ij} p_j' \]  

(14)

In the formula, \( N \) is the total number of nodes, \( u_j' \) and \( p_j' \) were displacement and surface forces of node.

The definition of influence coefficient (two occasions 2×2) for \( \mathbf{H}_{ij} = \sum_{j=1}^{N} \int_{\Gamma_j} p_j' \phi_i d\Gamma \)

\[ \mathbf{G}_{ij} = \sum_{j=1}^{N} \int_{\Gamma_j} u_j' \phi_i d\Gamma \]

In the formula, \( t \) is unit serial number that node number is \( j \), \( q \) is serial number of node \( j \) in unit \( t \). So we defined

\[ \mathbf{H}_{ij} = \begin{bmatrix} 
H_{ij} & i \neq j \\
H_{ij} + c' & i = j 
\end{bmatrix} \]

We get the formula (14) corresponding node \( i \)

\[ \sum_{j=1}^{N} \mathbf{H}_{ij} u_j' = \sum_{j=1}^{N} \mathbf{G}_{ij} p_j' \]

And so on, all nodes as the source point, we got the matrix equation (5).

Calculate the element of matrix \( H \) and \( G \) with the numerical integration formula, the calculation of the influence coefficients \( H_{ij} \) the \( G_{ij} \).

(1) Calculate sub matrix of \( HW(2 \times 6) \) and \( HW(2 \times 6) \), here the source point and 3 nodes is not a coincidence.

\[ HW = \int_{\Gamma_j} p_j' \phi d\Gamma = \int_{\Gamma_j} p_j' \phi [G] d\xi \]

\[ = \int_{\Gamma_j} \left( p_{11} p_{12} \right) \left( \phi_1 0 \phi_0 0 \phi_0 0 \right) [G] d\xi \]

\[ = \sum_{i=1}^{2} \phi_i \left( p_{11} p_{12} \right) \left( \phi_1 0 \phi_0 0 \phi_0 0 \right) [G] \]

\[ GW = \int_{\Gamma_j} u_j' \phi d\Gamma = \int_{\Gamma_j} u_j' \phi [K] d\xi \]

\[ = \int_{\Gamma_j} \left( u_{11} u_{12} \right) \left( \phi_1 0 \phi_0 0 \phi_0 0 \right) [K] d\xi \]

\[ = \sum_{i=1}^{4} \phi_i \left( u_{11} u_{12} \right) \left( \phi_1 0 \phi_0 0 \phi_0 0 \right) [G] \]

(2) treatment of singular integral

Considering Cartesian rigid body displacement of coordinate axis \( x_i \), the surface force and volume force must be zero. From equation (5) have

\[ H'I^s = 0 \]  

(17)

In the formula, \( I^s \) is unit length displacement group form \( q (q = 1,2,3 \) of all nodes), while the rest direction of the displacement is zero array. For Rigid body displacement in any direction, it can satisfy (17), we have

\[ H_{ij} = \sum_{j=1}^{N} H_{ij} \quad (i \neq j) \]  

(18)

According to the rigid body displacement principle, we can get the diagonal elements of block matrices for singular part of sub matrix \( HW(2 \times 6) \).

\[ H_{ij} = \sum_{j=1}^{N} H_{ij} \quad (i \neq j) \]  

(19)

In the formula, \( I \) is unit matrix for 2×2 (two occasions), \( H_{ij} \) can be indirectly got by the \( HW \) value added of formula (15) and (16). \( H \) is a square by
so the singularity occurs only in the diagonal elements, nonsingular part of submatrix $HW(2 \times 6)$ is still available calculation by formula (15).

Calculate sub matrix $GW(2 \times 6)$ by numerical integration method.

$$GW = \int_r u^T \phi d\Gamma = \int_r \begin{pmatrix} u_{i1}^T & u_{i2}^T & (\phi_1 & 0 & \phi_2 & 0 & \phi_1 & 0) \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_1 & 0 \end{pmatrix} d\Gamma$$

III. THE H-R ADAPTIVE BOUNDARY ELEMENT METHOD

A. Data Structure

(1) The data structure information array $klad(j, ii)$ of adaptive unit

$klad(j, ii) = IK + K_{PO}$

In the formula, $ii$ is serial number of the node $i$ on the adaptive unit; $j$ is the segmentation process counter, which records the initial element subdivision number and class level; $IK$ record number of new added nodes in subdivision; when $K_{PO}$ is 0, 1 or 2, nodes $i$ is not adaptive, new node is in left or right of the node $i$.

(2) Adaptive subdivision information array $ID(i)$, $IP$ of element node $i$

$$ID(i)=\begin{cases} 0 \rightarrow IP = 0 & \rightarrow j = j, \\ IK = IK, & (K_{PO} = 0) \\ 1 \rightarrow IP = 1 & \rightarrow J = j + 1, \\ IK = IK + 2, & (K_{PO} = 1, 2) \end{cases}$$

Each subdivision, array $klad(j, ii)$, $IK$, $j$ only to change some elements, and adaptive subdivision information $ID(i)$, $IP$ of node must be refreshed to form a new array, in order to reflect the node adaptive information change.

As shown in Figure 3, initial unit integral (with 8 node unit as an example) defined as unit, the adaptive integral called, is the original parent unit into 4 smaller units through the refinement (known as a subsidiary unit), each unit with its parent unit completely similar, then transfer the singular integral of initial parent unit to 4 small unit, so as to obtain a first refinement decomposition, so that the completion of the first refinement of the initial parent unit (as shown in Figure 4). Each refinement will receive a detailed decomposition, in order to ensure the accuracy of integral in the given range of convergence, the need for multilevel refinement to the initial father unit (Fig. 5-6). Multilevel refinement process in the unit, each unit only in the fine points of this layer, when this layer integral is completed, the unit automatically clear, so that the refining unit for “temporary unit”, it only contributes to find precise integration, does not increase the total number of boundary element mesh.

B. Calculation of H-R Adaptive Boundary Element Method

The H-R adaptive boundary element can be represented in Figure 7, which mainly consists of the following three parts:

- **error estimation**: Analyze and estimate discretization error of the boundary element approximate solution
- **adaptive subdivision strategy**: used to judge boundary element solution whether meet the accuracy requirement, and find out the units which do not meet the precision requirements for the next step.

![Figure 3. Initial element](image3.png)

![Figure 4. Subelements after first subdivision](image4.png)

![Figure 5. Subelements after second subdivision](image5.png)

![Figure 6. Subelements after final subdivision](image6.png)

![Figure 7. Calculation flow chart of H-R adaptive boundary element method](image7.png)
C. Interpolation Error Calculation

In the boundary integral equation, residual function is zero at the boundary nodes, due to the discretization error, the boundary element solution is not equal to the true solution. For the boundary variables \( \Psi_i \) (displacement or stress) if interpolation expression of approximate solution and the interpolation expression of real solution were recorded as \( \tilde{\Psi}_i \) and \( \tilde{\Psi}_i \), and noticed that they were using the same shape functions; boundary variables, the error of the approximate solution can be expressed as:

\[
e_{\nu_i} = \Psi_i - \tilde{\Psi}_i = \left( \tilde{\Psi}_i - \tilde{\Psi}_i \right) + \sum_e (N_i(\zeta) e_{\nu_i}) \tag{21}\]

In the formula, \( \{\tilde{\Psi}_i - \tilde{\Psi}_i\} \) is the interpolation error caused by the real solution after interpolation, \( e_{\nu_i} \) is the pointwise error at the nodes of the approximate solution and the true solution for the boundary variables \( \Psi_i \) (displacement or stress).

When substitute the approximate solution \( \tilde{u}_i \) and \( \tilde{p}_i \) into type (1), unless the source point x is the boundary node, otherwise the boundary integral equation is generally not satisfied, you can define a "residual displacement" \( R^R_i(x) \) to represent the [9]:

\[
R^R_i(x) = c_i(x) \left[ u_i(x) - \tilde{u}_i(x) \right] + \int p_i^* (u_i - \tilde{u}_i) d\Gamma - \int u_i^* (p_i - \tilde{p}_i) d\Gamma \tag{22}\]

Put the formula (5) into type (6), interpolation expression of the real solution \( u_i \) and \( p_i \) were recorded as \( \bar{u}_i \) and \( \bar{p}_i \):

\[
R^R_i(x) = c_i(x) \left[ u_i(x) - \bar{u}_i(x) \right] + \int p_i^* (u_i - \bar{u}_i) d\Gamma - \int u_i^* (p_i - \bar{p}_i) d\Gamma \tag{23}\]

The above type can also be written as:

\[
R^R_i(x) = R^m_i(x) + R^{node}_i(x) \tag{24}\]

\( R^{node}_i(x) \) is called the node residual, it generated because the node value difference of the real solution and the approximate solution:

\[
R^{node}_i(x) = c_i(x) \left[ \bar{u}_i(x) - \tilde{u}_i(x) \right] + \int p_i^* (\bar{u}_i - \tilde{u}_i) d\Gamma - \int u_i^* (\bar{p}_i - \tilde{p}_i) d\Gamma \tag{26}\]

From the type (6), residual error \( R^R_i(x) \) is zero at the boundary nodes, i.e.:

\[
R^R_i(x) + R^{node}_i(x) = 0 \tag{27}\]

Reference type (5), the type (11) can be written as:

\[
c_i(x) e_{\nu_i}(x) + \sum_{j=1}^{N_e} \left[ \int p_j^* N_j (\zeta) d\Gamma \ast e_{\nu_i} \right] - \sum_{j=1}^{N_e} \sum_{k=1}^{N_p} \left[ \int u_k^* N_j (\zeta) d\Gamma \ast e_{\nu_i} \right] = -R^m_i(x) \tag{28}\]

The above formula can also be written in matrix form:

\[
[A] e' = -R^m \tag{29}\]

In the formula, \( [A] \) is the coefficient matrix, it is equal to the coefficient matrix of the collocation method; \( \{e'\} \) is node error vector of boundary unknown quantity.

Type (29) gives the relationship of the pointwise error estimates between the interpolation residual vector and the boundary element solution, the pointwise error of the boundary element solution and estimation interpolation residue are linked. In theory, if we know the interpolation residual vector \( \{R^m\} \), by solving the linear equation of formula (29) we can get the pointwise error of boundary element solution at the nodes.

However, In fact the real solution is always unknown, residual vector can only get by approximate method; then if using type (29) directly to calculate pointwise error, it will be affected by the stability of coefficient matrix. But the stability of coefficient matrix in boundary element equations is not good; the pointwise error estimates got by type (29) may have large distortion. Therefore, the following discussion is that using the interpolation residual vector indirectly to estimate pointwise error, this can avoid the difficulty due to the stability of system matrix.

IV. ERROR ESTIMATION

Residual analysis is one of the best error estimation method, the reason is that it need not know the exact solutions of displacement and stress, and the displacement error and traction error of boundary element can be linked to by the boundary integral equation, it is relatively easy to calculate. In the H-R adaptive boundary element process, the interpolation error of the ith times iteration can be represent by the difference of approximated solution of ith time iterative and the
approximate solution of the \( i-1 \) time iteration proceeds, i.e.:

\[
\begin{align*}
    u_k &= \Pi u_k \approx \tilde{u}_k - \tilde{u}_k^{-1} \\
    p_k &= \Pi p_k \approx \tilde{p}_k - \tilde{p}_k^{-1}.
\end{align*}
\]  

In the formula, \( \tilde{u}_k \), \( \tilde{u}_k^{-1} \) respectively represent approximate solution of \( i \)th time and the \( i-1 \) time iterative, \( \Pi u_k \) is the interpolation function of the boundary element solution; Surface force is similar.

The interpolation residual vector \( R_{ik}^m (x_i) \) can be expressed as

\[
R_{ik}^m (x_i) = \int_{\Gamma} p_k^m (u_k - \Pi u_k) d\Gamma - \int_{\Gamma} u_k^m (p_k - \Pi p_k) d\Gamma
\]

\[
= \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} p_k^m (u_k - \Pi u_k) d\Gamma - \int_{\Gamma_j} u_k^m (p_k - \Pi p_k) d\Gamma \right\}
\]

\[
\approx \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} p_k^m (\tilde{u}_k - \tilde{u}_k^{-1}) d\Gamma - \int_{\Gamma_j} u_k^m (\tilde{p}_k - \tilde{p}_k^{-1}) d\Gamma \right\}
\]

\[
\approx \sum_{j=1}^{NE} (R_k)_{ij}.
\]

In the formula, \( (R_k)_{ij} \) is the \( k \)th component of interpolation residuals in the unit \( j \) when the loading point is \( x_i \).

Error indicators \( \lambda_{j} \), of unit \( j \) can be defined as

\[
\lambda_{j} = \max \left( \| (R_k)_{ij} \|, \| (R_k)_{ij} \| \right), i=1,2,\ldots,N \]  

(32)

Residual vector \( R \) can be measure by the following modules

\[
\| R \| = \| R_{ik}^m (x_i) \| + \| R_{ik}^m (x_i) \| + \cdots + \| R_{ik}^m (x_i) \|
\]

\[
= \sum_{j=1}^{NE} \left\{ \sum_{i=1}^{N} (R_k)_{ij} \right\} + \sum_{i=1}^{N} (R_k)_{ij} \|
\]

\[
\leq \sum_{j=1}^{NE} \sum_{i=1}^{N} \left\{ (R_k)_{ij} \right\} + \| (R_k)_{ij} \| \leq \sum_{j=1}^{NE} \lambda_{j}.
\]

(33)

Therefore, the overall error estimator \( \eta \) can be defined as

\[
\eta = \| R \| = \sum_{j=1}^{NE} \lambda_{j}.
\]

V. NUMERICAL RESULTS

Example 1 consider thin cantilever beam boundary element solutions and error distribution along the entire boundary under uniform lateral load.

Use adaptive boundary element method to solve the cantilever beam bearing transverse load distribution

The H-R adaptive boundary element methods in this paper, except the region near the fixed end corner nodes, error distribution along almost the entire boundary was obtained. In order to calculate distribution of error near the area given traction boundary adjacent unit public nodes, we use the small arc to remove the boundary node that the radius is \( \varepsilon = 0.025\Delta \). For the sake of convenience, we think the public node adjacent units adjacent region on the surface force is uniform distribution.

The example is used to check up the effectiveness of the adaptive boundary element method. According to the adaptive method for grid, boundary will be divided into 14 quadratic isoparametric element, a direct error estimation is \( \Delta = 0.07 \). If we take the permission error \( \Delta = 0.0101 \), then to improve the unit is nine. If the error \( \Delta \) we take license=0.0101, so the unit improved is 9. In accordance with the adaptive boundary element method, the boundary is divided into 21 quadratic isoparametric elements. When the boundary is divided into 25 quadratic isoparametric elements, error estimate is 0.01003, and it meets the given requirements, so the whole process termination. The estimation of direct error determined for each stage is presented in table 1.

![Adaptive mesh subdivision process](image)

**Figure 9. Adaptive mesh subdivision process**

**Table 1. The estimation of direct error determined for each stage**

<table>
<thead>
<tr>
<th>Stage</th>
<th>The number of units</th>
<th>Direct error estimation $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>0.05842</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
<td>0.01102</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>0.011003</td>
</tr>
</tbody>
</table>

With the increase in the number of iterations, the local error indicating factor increases on the unit which is not subdivided, the local error indicating factor decreases on subdivided unit, the local error of the indicator on each
unit distribution tends to be uniform, the grid becomes optimization.

Example 2 Elastic solution of thick wall cylinder subjected to internal pressure (Figure 10).

The geometric size of thick wall cylinder is: the inner radius $R_1 = 3.0$, outer radius $R_2 = 6.0$, height $H = 1.0$, load $P = 1.0$, material parameters: $E = 1.0$, $v = 0.3$.

Figure 10. The geometric model of thick wall cylinder under internal pressure

### TABLE II. ELASTIC SOLUTION OF THICK WALL CYLINDER SUBJECTED TO INTERNAL PRESSURE

<table>
<thead>
<tr>
<th>nodes</th>
<th>$R$</th>
<th>$h_1$</th>
<th>Analytical solution</th>
<th>solution of this paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0</td>
<td>5.9000</td>
<td>5.9000</td>
<td></td>
</tr>
<tr>
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</tr>
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</tr>
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<td>6.0</td>
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<td>3.9998</td>
<td></td>
</tr>
</tbody>
</table>

VI. CONCLUSIONS

This paper presents a new solution error estimation method and it is suitable for H-R adaptive boundary element subdivision process, this method analyze the interpolation error of variables in H-R adaptive subdivision of discrete grid, and use it as the basis of boundary element error estimation. Because the mesh gradient may generate a large change in the process of H-R adaptive subdivision, the system matrix is not stable; and in the method, the pointwise error of boundary element solution is indirectly estimated by interpolation error, thus avoiding the analysis distortion caused by instability of the system matrices. Its effectiveness is verified by example.

The proposed error estimation method has the following characteristics:

1. The method is the residuals analysis method, it is convenient for elastostatic problems which can be easily applied to both displacement and traction boundary conditions;
2. The method is a posteriori estimation, error of iteration grid estimates are obtained by comparing the results of the previous iteration;
3. The method is a kind of asymptotic method, the overall error estimator tends to zero with the geometry size units decreases.

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REFERENCES


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