A Novel Inpainting Model for Partial Differential Equation Based on Curvature Function

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Abstract—For the curvature function of Curvature-Driven Diffusions model (CDD), this paper studies the impact of the CDD on the inpainting, and a novel inpainting model based on partial differential equation (PDE) is proposed which contains five sub-models. For simplification, a discrete model based on this PDE is given. Experimental results show that we can obtain better inpainting effects compared with each other from those sub-models.

Index Terms—Digital Image Inpainting, Partial Differential Equations (PDE), Curvature

I. INTRODUCTION

Image inpainting can fill up the defect field of image in accordance with certain rules of procedure using information of image, which improve the visualization performance of the original image. In mathematics, two models can be used to describe an inpainting image, for example, Data model (Read existing information) and Prior model (Gets the type original image). In this way, we can take an inpainting problem as optimization problem for an energy function which using mathematics language to express the progressing of image inpainting. Then, we can obtain a data model and a prior model of the image after inpainting[1]. This kind of algorithm always called image inpainting algorithm based on PDE. Another kind of image completion algorithm is mainly used to large objects which may loss of information[2]. Its main algorithm based on texture synthesis, the main idea is to select an element from the boundary of inpainting field firstly and on the element, which can choose the appropriate size of texture objects based on its features[3, 4]. Finally we can find the most relatively texture around the inpainting field for it and instead of it secondly.

Bertalmio-Sapiro-Caselles-Ballester (BSCB) model has been raised in literature[5], which is a third-order PDE, its main idea is inpainting along the direction of illumination (means the grayscale image in the same level consisting of a series of points on a line) iterations which considering the information around inpainting field to its inside, then, calculate the discrete gradient vector (means spatial maximum change of direction) of each point stood on edge line of inpainting field with rotating $90^\circ$. This model can be better inpaint image and avoid edge blur, but it leak on stability and poor outcomes represents which is not good for a rich texture image.

Literature[6] proposed a total variation algorithm (TV model) and is a second-order PDE, during the progress of image inpainting, it will make anisotropic diffusion. An advantage of this model is that it can keep edge and guarantee value simply. We know that it comes from the p-Laplace[7] inpainting model, and when the iteration $P$ had been set as a stable value, we can name it as the TV model. Although it works better than the BSCB model about stability and robustness. It may break the theory of visual connection and causes ladder-effect.

Literature [8, 9, 10] improved TV inpainting model in the domestic, these methods make the inpainting effect better than the ordinary TV model. As we know that the most difficult problem of TV model is that it may make staircase effect, which influence our inpainting performance in some extends.

A Curvature-Driven Diffusion (CDD) model has been raised in Literature [11], which is a third-order PDE simulation of diffusion and spreading process. Although it can keep a well inpainting performance, its higher computational complexity makes this model poor because of its relation to the anisotropic diffusion.

Literature[12] locates about fast algorithm of CDD model, it aims to accelerate inpainting time of the traditional CDD model.

Literature[13] puts forward a harmonic inpainting model, an isotropic diffusion with second-order PDE, so it can exclude noise while obscures edge.

Literature[14] submits a p-harmonic model based on literatures [6, 12], they combine the advantages of the TV model and the harmonic model which shorten the inpainting computing complexity and better the inpainting performance.
Literature\cite{15} proposes a Criminisi inpainting algorithm, it belongs to the second type of inpainting technology, which is a sample-based texture synthesis algorithm not only image with large bulk of loss information can be repaired, but also details, both in time and visual effects better than other algorithms.

II. BRIEF INTRODUCE FOR TV MODEL AND CDD MODELS

A. TV model

The total variation(\(TV\)) model was shown by Tony Chan in 2002, the main idea of the TV model is that it can keep the edge while anisotropic diffusion to obtain a result which can restore all the images. From the equation defined from \(P\)-laplace equation

\[
\Delta_p l - \lambda (I - I^0) = 0, 1 < p < \infty,
\]

where \(I^0 = I + N\) is a model contains an additive noise and the relatively energy functional equation is:

\[
J[I] = \frac{1}{p} \int_\Omega | \nabla I |^p d\sigma + \frac{\lambda}{2} \int_{\Omega \setminus D} (I - I^0)^2 d\sigma.
\]

Especially when \(p = 1\), we can regard the image as a smooth piecewise function which is good for establishing models for images on bounded space, then the total variation image inpainting model is:

\[
\min J[I] = \int_\Omega | \nabla I | d\sigma + \frac{\lambda}{2} \int_{\Omega \setminus D} (I - I^0)^2 d\sigma
\]  

where \((x, y)\) is the image grayscale function, reduced it to \(I\), \(I^0\) is the defect image grayscale function, \(D\) is the inpainting field, \(\Omega\) is the total image field. \(\Omega \setminus D\) is a known image field, \(\lambda\) is the multiplier of Lagrange, the above formula is a double integration, \(d\sigma = dxdy\).

Based on the variation theory\cite{16}, we can determine the Euler equations as follows:

\[
-\nabla \frac{\nabla I}{| \nabla I |} + \lambda_D(x, y) \cdot (I - I^0) = 0
\]  

where

\[
\lambda_D(x, y) = \frac{\lambda}{| \nabla I |} \Omega \setminus D(x, y) = \begin{cases} \lambda, & (x, y) \in \Omega \setminus D, \\ 0, & (x, y) \in D, \end{cases}
\]

and

\[
I_{\Omega \setminus D}(x, y) = \begin{cases} 1, & (x, y) \in \Omega \setminus D, \\ 0, & (x, y) \in D, \end{cases}
\]

We should consider that \(| \nabla I |\) may be near to 0, so we make the \(\nu = \frac{1}{\sqrt{| \nabla I |^2 + \epsilon^2}}\) instead of \(\nu = \frac{1}{| \nabla I |}\), while the \(\nu = \frac{1}{| \nabla I |}\) is a small positive number, which makes the strength of diffusion more huge while grads be small and be small while grads be huge.

Euler showed a power definition of curve firstly in research under the action of external forces, we have already know two end point of curve A and B, the strength is \(d\), the equation is \(E(d) = \int_a^b (\alpha + \beta k^2) ds\), Chan and his team elastic energy into the TV model, then TV-Euler had been showed:

\[
E[I | I^0, D] = \int_\Omega \phi(k) | \nabla I | d\sigma + \frac{\lambda}{2} \int_{\Omega \setminus D} (I - I^0)^2 d\sigma,
\]

where \(\phi(k) = \alpha + \beta k^2\), \(k = \sin (\frac{\nabla I}{| \nabla I |})\), when \(\alpha = 1, \beta = 0\), this is a TV model.

B. CDD model

In order to make TV model keeps a better visual connectivity, we modified it and considered a CDD model. Modifying the diffusion coefficients of TV model \(v = \frac{1}{| \nabla I |}\) to \(v = \frac{1}{g(\nu)}\), where

\[
g(\nu) = \begin{cases} 0, & k = 0, \\ \nu, & 0 < k < \infty. \end{cases}
\]

In generally, we suppose \(g(s) = s^a, s > 0, a \geq 1\), while we set \(a = 1\) here, because it makes diffusion be strong in large curvature while weaker in small curvature, then the corresponding CDD model is:

\[
\frac{\partial l}{\partial t} = \nabla \left[ \frac{g(\nu| |\nabla I|)}{| |\nabla I| |} \right] + \lambda_o (I^0 - l)
\]

\[
g(\nu) \text {is increasing function,} k = \sin \left(\frac{\nabla I}{| |\nabla I| |}\right)
\]

This is better than TV model of the inpainting performance.

C. P-harmonic model

We can establish a relative energy function model according to the noise included in the image when \(1 < p < 2\) as follows:

(1) Model I(no noise)

\[
J[I] = \int_\Omega \frac{1}{p} | |\nabla I| |^p d\sigma
\]

(2) Model II(noise)

\[
J[I] = \int_\Omega \frac{1}{p} | |\nabla I| |^p d\sigma + \frac{\lambda_o}{2} \int_{\Omega \setminus D} (I - I^0)^2 d\sigma
\]

\[
\lambda_o(x, y) = \lambda (x, y) \in \Omega \setminus D
\]

\[
0, \quad (x, y) \in D
\]

In general, we discuss the noise model, the energy functional is:
After inpainting the image, we can get the value from the following functions:

\[
\frac{\partial l}{\partial t} = \nabla \cdot \left( \left| \nabla I \right|^p \nabla I \right) + \lambda_0 (t^0 - I) \quad \text{innerD}
\]

\[
l = t^0 \quad \text{innerD}
\]

III. REALIZATION OF THE PARTIAL DIFFERENTIAL INPAINTING MODEL BASED ON CDD MODEL

A. Model proposed

Based on the foundation of the TV model and the CDD model, we can study the influence of the curvature function \( g(s) \) on inpainting performance, here, we construct the curvature function as follows:

\[
g(s) = a s^p + b (e^c - 1) + c \ln(s + 1) + d, \quad p \geq 1, s > 0, \quad a, b, c, d \geq 0.
\]

The purpose of such a construction, firstly, it reaches to discuss the influence for the result of inpainting of these different curvature functions and which contains power functions, exponential functions, logarithmic functions and those functional images are shown in figure 1; Secondly, it summarizes many models which have been introduced before, such as \( d = 1 \), while others are 0, back into TV model; \( a = 1, p = 1 \), while others coefficient are 0, obtains a general CDD model[17]. And to some extent, it can supply a gap for TV model during inpainting a large square of defects field which may destroy visual connectivities. So the new established inpainting model is

\[
\frac{\partial l}{\partial t} = \nabla \cdot \left( \left| \nabla I \right|^p \nabla I \right) + \lambda_0 (t^0 - I) \quad \text{innerD}
\]

\[
g(s) = a s^p + b (e^c - 1) + c \ln(s + 1) + d, \quad p \geq 1, s > 0
\]

B. Discretization of Model

For calculate conveniently, we need to discrete the above models and put forward discrete model. After discretizing a serials of PDE, the corresponding digital filter has been produced through a digital iteration method, we consult from literature[6,12] to use half-point difference scheme method that is considering for the gradient solutions algorithm of eight neighborhoods of inpainting pixel point.

As shown in figure 2, \( O \) is an inpainting pixel point, \( \wedge = \{N, S, W, E\} \) is \( O \)'s four neighborhood collections. \( \wedge = \{n, s, w, e\} \) is been used for four half-pixel neighborhoods.

Ordering \( v = (v', v^x) = g([k || |V| V]) \), then give a central difference numerical value for divergence:

\[
V \cdot V = \frac{\partial v^x}{\partial x} + \frac{\partial v^y}{\partial y} = \frac{v^x - v'^x}{h} + \frac{v^y - v'^y}{h}
\]

here, we can set step \( h = 1 \).

In order to avoid \( |\nabla I| \) to be 0, we can set parameters as following:

\[
|\nabla I| = \sqrt{v^2 + |\nabla I|^2},
\]

or

\[
|\nabla I| = \varepsilon + |\nabla I|.
\]

Then we calculate the similarity of \( v^x, v^y, v'^x, v'^y \):
In order to get the results of the above expression, we still need to calculate 
\[ \nabla I_x, \nabla I_y \] , (| |), (| |), (| |), (| |), (| |), (| |), (| |). 
When 
\[ g(k) = 1 \] , we can get the computational formula 
\[ \nabla I_x, \nabla I_y \] = \[ \nabla I_x, \nabla I_y \] . If the value of \( g(\cdot) \) has given, we can obtain the value of \( g(k) \) . Apparently, the inpainting value of the target pixel point \( O \) just relates to 33 \( \times \) 3 pixel points of its neighborhood (which did not mark out the half-pixels in figure), which has been shown in table 1:

TABLE 1. PIXEL POINT \( O \)'S 3\( \times \)3 NEIGHBORHOOD

<table>
<thead>
<tr>
<th>NW</th>
<th>N</th>
<th>NE</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>O</td>
<td>E</td>
</tr>
<tr>
<td>SW</td>
<td>S</td>
<td>SE</td>
</tr>
</tbody>
</table>

In this way, we need to calculate the value of \( k_q \), where \( q \) is the half-pixel point of \( \wedge \), which we also choose difference method to calculate it. Assuming that:

\[ \nabla I_x, \nabla I_y \] = \[ \nabla I_x, \nabla I_y \] \( \nabla I_x, \nabla I_y \) = \( (v_1, v_2) \),

\[ k_q = \nabla \frac{\nabla I_x, \nabla I_y}{|\nabla I_x, \nabla I_y|} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}. \]

The calculation formula of \( k_q \) is given in the following formulas. We expand it as follows:

\[ k = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{v_1^{\text{E}} - v_1^{\text{O}} + v_1^{\text{NO}} - v_1^{\text{SE}} + v_2^{\text{E}} - v_2^{\text{S}}}{4h} \]

But if we can’t calculate \( v_1^{\text{E}} \) based on the formula in the 3\( \times \)3 neighborhood, then we may turn to consider the 5\( \times \)5 neighborhood, which are shown in table 2 and figure 3:

TABLE 2. PIXEL POINT \( O \)'S 5\( \times \)5 NEIGHBORHOOD

<table>
<thead>
<tr>
<th>NW</th>
<th>N</th>
<th>NE</th>
<th>O_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>O</td>
<td>E</td>
<td>O_2</td>
</tr>
</tbody>
</table>

In the same way, we can obtain the values of \( v_1^{\text{E}}, v_2^{\text{E}}, v_1^{\text{NE}}, v_2^{\text{NE}} \). They are as follows:

\[
\begin{align*}
\nabla (g(k)|I|) &= \nabla v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \\
&= \frac{v_1 - v_1^{\text{O}} + v_1^{\text{NO}} - v_1^{\text{SE}} + v_2^{\text{E}} - v_2^{\text{S}}}{h} \\
&+ \sum_{q_1, q_2, q_3} g(k) |I_q - I_o| 
\end{align*}
\]

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where \( Q \in \wedge, q \in \wedge \) is to say that \( Q \) is the pixel point in \( \wedge \), \( q \) is the corresponding half-pixel point of \( \wedge \), if \( Q \) equal to \( E \), then \( q \) equal to \( e \). After setting the above formula to formula (4), we can obtain:

\[
I(O) = \sum_{Q \in \wedge, q \in \wedge} \frac{g \left( k_{ij} \right)}{V_{ij}^{k}} h_{ij} I(Q) + \sum_{Q \in \wedge, q \in \wedge} \frac{g \left( k_{ij} \right)}{V_{ij}^{k}} f(Q)
\]

Then, we reformulated it as follows:

\[
q_{ij} \frac{g \left( k_{ij} \right)}{V_{ij}^{k}} h_{ij} = \frac{q_{ij}}{\lambda_{ij}(O) + \sum_{Q \in \wedge, q \in \wedge} \frac{g \left( k_{ij} \right)}{V_{ij}^{k}}}
\]

Finally, we can obtain:

\[
I(O) = \sum_{Q \in \wedge, q \in \wedge} h_{ij} I(Q) + h_{ij} I^{\lambda}(O)
\]

The above expression can be understood as inpainting target pixel values through doing weighted \( h_{ij} \) for its neighborhood \( I(Q) \), so it can be considered as weighted inpainting model\(^{[10]}\).

Using Gauss-Jacobin iterative method, where \( n \) represents its iterations, the above formula can be conveyed as:

\[
I(O) = \sum_{Q \in \wedge, q \in \wedge} h_{ij}^{n} I^{n}(Q) + h_{ij}^{n} I^{\lambda}(O)
\]

While in the equation \( \sum_{Q \in \wedge, q \in \wedge} h_{ij} + h_{ij} = 1 \).

### C. Model simulation

When progresses with numerical simulation, ordinary we take mask to convince the inpainting field \( D \), then use inpainting algorithm automatically to restore those missing information according to the around information of inpainting field. Depend on the formula (III.10), the steps of improved model for differential image inpainting algorithm are:

1. read image \( I \) and mask information \( I_{m} \) into computer, give the value of \( \lambda, \varepsilon \);
2. performing (3),(4),(5) on each pixel of mask, until reaching the iterations that in rules;
3. if pixel locates beyond inpainting field, \( \lambda_{ij} = \lambda \), otherwise \( \lambda = 0 \);
4. depend on formula (9) to calculate \( \omega_{ij}, h_{ij}, h_{ij} \);
5. get a new pixel according to the (10) and store it to new image\(^{[10]}\).

In the followings simulation test, \( \lambda = 0, \varepsilon = 0.0001 \). We simulate using MATLAB 7.0, some basically procession of image have been related to literature\(^{[12]}\).

Test I  The comparison chart of visual effect on each sub-model with stunt making and data evaluation.

Through making stunt for image we can wipe out steel wire in figure 4. Firstly labeling that means using mask to label the steel wire, as has shown in A, this paper have labeled it by pure white color(255).The inpainting performance will be better if use impure white color and the larger the iterations, the better effect of inpainting.

| TABLE 3. PSNR THE QUALITY EVALUATION METRIC OF EACH SUB-MODEL AND ITERATION DATA COUNT TABLE |
|---------------------------------|-------|-------|-------|
| Constant CM(TV)                | N=10  | N=20  | N=40  |
| Power CM(CDD)                  | 63.412| 74.178| 75.735|
| Exponential CM                 | 53.627| 56.932| 64.951|
| Logarithmic CM                 | 54.214| 59.132| 69.692|
| Comprehensive CM               | 53.312| 55.343| 61.986|
| N=60                           | N=80  | N=100 |
| Constant CM(TV)                | 75.799| 75.832| 75.819|
| Power CM(CDD)                  | 71.060| 74.189| 74.350|
| Exponential CM                 | 61.686| 66.566| 70.706|
| Logarithmic CM                 | 74.739| 74.732| 74.812|
| Comprehensive CM               | 68.824| 73.339| 74.114|

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From the table 3 we know that inpainting performance is the best for the TV model when deal with these scores in small square fields, then logarithm curvature model, power curvature model, comprehensive curvature model and exponential curvature model. We can conclude that rather the indigestion of $g(t)$ have not represent its advantage for us than the faster it increases the worst the inpainting effect when dealing with smooth and small square fields in the image inpainting.

Figure 6 is the image of table 3, from top to bottom are the images relate to the TV model, logarithm curvature model, CDD model, comprehensive curvature model and exponential curvature model. While the PSNR is a peak value signal noise ratio, which is used for taking measure for the inpainting images, the function is:

$$PSNR = 10 \cdot \log \left( \frac{255^2}{\frac{1}{MN} \sum_{x=1}^{M} \sum_{y=1}^{N} [I(x,y) - \hat{I}(x,y)]^2} \right)$$

This is better than the usually measures such as SNR:

$$SNR = 10 \cdot \log \left( \frac{\sum_{x=1}^{M} \sum_{y=1}^{N} \hat{I}(x,y)^2}{\sum_{x=1}^{M} \sum_{y=1}^{N} [I(x,y) - \hat{I}(x,y)]^2} \right)$$

The PSNR is a normal measures to use in inpainting images, especially used for some images need to be wiped noise while the bigger the value of it, the better of the effect[20].

Test II. Comparison chart of multicolor image inpainting effect on each sub-model

The inpainting field of the test which is a red field in a little large square in the multicolor image B, color is [255, 0, 0], from the inpainting effect we know that each model represents its effect slowly. Exponential curvature model still has a part of black field not repaired and TV model, logarithm curvature model, CDD model and comprehensive model have the better inpainting performance.

Test3 Image denoising test for each sub-model

The noise plus in the test denoising image is impulsive noise, and the impulsive probability is 10%, iteration is 2. We know from the result of the test that TV model makes work very functional especially on denoising but it makes a soft edge for image. Yet, logarithm curvature model, exponential curvature model and comprehensive model are not better than the TV model on denoising effect, but they still keep the image clear[21].
Test 4 Inpainting image and denoising in the same time for logarithm curvature of sub-model

Test 6 Test of removing text by logarithm curvature model of sub-model

The image which had been pulsed words to the gray image for the middle image of the above images, removing its words on the surface and the color is \([255, 255, 0]\), if we wipe out some white dot for the image, the new image will looks like removed back to the original image from visual terms. So we know that the logarithm curvature model has a good effect of removing text[5].

D. Conclusion

Based on the TV model and the CDD model in the II, we had modified it according to the advantage and disadvantage of the TV model, we gived out a continual partial diffusion equation model, when all iterations of logarithm function in model have been set differently, we get some other sub-models, which contains those models: TV model, CDD model, exponential curvature model, logarithm curvature model and comprehensive model. For simplicity to study, we put forward discretization model in the whole paper, we considered the area for discretizing it in a \(5 \times 5\) area. So we can obtain the related discretization model for each sub-model, and the new TV model and the CDD model after discretizing are different while used with the means of difference equation and difference area are different. At last we have simulated it by MATLAB, we test a set of images, then we could convince that discretization model has a well inpainting effect on inpainting images and denoising for small area, and these sub-models are well except exponential curvature model.

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